HD–TVP–95–13

One Loop Integrals at Finite Temperature and Density

P. Rehberg* and S. P. Klevansky
Institut für Theoretische Physik, Universität Heidelberg,
Philosophenweg 19, D–69120 Heidelberg, Germany

(September 1995)

Abstract

The technique of decomposing Feynman diagrams at the one loop level into elementary integrals is generalized to the imaginary time Matsubara formalism. The three lowest integrals, containing one, two and three fermion lines, are provided in a form that separates out the real and imaginary parts of these complex functions, according to the input arguments, in a fashion that is suitable for numerical evaluation. The forms given can be evaluated for arbitrary values of temperature, particle mass, particle momenta and chemical potential.

MSC numbers: 65K05, 81Q30, 81T18, 82B10

Typeset using REVTEX

*E-Mail: `Peter@Frodo.TPhys.Uni-Heidelberg.DE`

I. INTRODUCTION

The imaginary time Matsubara formalism for Green functions [1] is a venerated method of dealing with problems of finite temperature. This is so because of the fact that Wick's theorem can easily be shown to hold for the Matsubara operators. As such, this method of dealing with finite temperature systems has found wide application, primarily in condensed matter physics [2], but also recently in problems relevant for high energy physics. In particular, it has been recently used in studies of effective QCD Lagrangians (see, e. g. the reviews [3] and [4] and references cited therein), through which an understanding of hot and dense matter is sought [5–9]. However, to date most such studies have often been constrained by specific parameter choices to enable tractability – often degenerate particle masses, zero chemical potential or special kinematics are chosen. Furthermore, often only the principal values of these integrals are considered, with the complex nature being completely ignored. An extension of those calculations to physically relevant cases with arbitrary parameters and generalized kinematics, as may be appropriate for example, in calculating cross sections in an $SU(2)$ or $SU(3)$ effective model of QCD [7–9], becomes technically extremely complicated. It is thus the purpose of this paper to address the technical problems associated with such integrals, and present them in a form suitable for the numerical evaluation of both their real and imaginary components. This work has come about through a study of hadronization processes within the Nambu–Jona–Lasinio model [9], and is intended to provide the technical knowledge required for such functions for researchers who are active in this field. The associated computer code conformal to the analysis presented here is freely available[1].

For calculations at zero temperature and density, it has been shown [10], that all occurring transition amplitudes at the one loop level can be decomposed into a number of elementary integrals, which in turn can be classified by the number of particle lines they contain. Since this decomposition technique can be generalized to the Matsubara formalism, the aim of this paper is to provide suitable forms for

[1]Programs are available by anonymous ftp to `Trick.MPI-HD.MPG.DE`, the location of the archive file is `/pub/loopies-1.0.shar.Z`.

the elementary integrals containing one, two and three fermion lines, which we term A, B_0 and C_0, respectively, which can be used for numerical evaluation. (The calculation can be easily taken over to boson loops on replacing the Fermi distribution by the Bose distribution.) We analyze these integrals, explicitly displaying their complex nature that is a function of the input parameters. The parameters that may then be specified can be of the most general form, i. e. arbitrary temperatures, particle masses, chemical potentials and kinematics. As such, the associated computer code can be treated as a library–like 'black box'.

We note at this point that the calculation of the one loop integral A is straightforward. Its evaluation here serves merely to remind the reader of the usage of the Matsubara formalism and it also renders this work complete. The evaluation of B_0, on the other hand, serves to demonstrate the complexities involved in a consistent evaluation of this function. Finally we deal with the three line integral C_0, which then is highly nontrivial and complicated.

It is shown in this paper that each of integrals B_0 and C_0 (and thus implicitly all higher one loop integrals) can be written as a sum over terms that are constructed from a single generic function that is taken at different values of its arguments. The analytical structure of these generic functions is connected with the number of nonvanishing external momenta. We are thus required to distinguish between several cases in evaluating the generic functions when constructing a physical quantity.

As is usually the case in relativistic quantum mechanics, the one and two fermion line integrals A and B_0 are divergent, which means that they have to be regularized. We do this using a three momentum cutoff Λ, which means that we integrate over all momenta with $|\vec{p}| < \Lambda$. The three fermion line integral C_0 is however convergent. We nevertheless perform our analysis of this function including a three momentum cutoff. The reasons for doing this are twofold: (i) the extension to the case $\Lambda \to \infty$ is simple to obtain and provides a check of our calculations, and (ii) for models such as that of Nambu and Jona–Lasinio extended to QCD, it is regarded as internally consistent within the model to restrict all quark momenta in all loops.

Having dealt with the analytical complexities of the loop integrals, we demonstrate in the last section the usage of the integral decomposition in a simple physical example, viz. that of calculating the kaon masses in the $SU(3)$ Nambu–Jona–Lasinio model. This involves the integrals A and B_0. Use of C_0 may be found, for example,

in Ref. [9].

This paper is organized as follows: in Sec. II, we define the functions A, B_0 and C_0. These functions are brought into a form suitable for numerical integration in Secs. III, IV and V, respectively. At the end of Sec. IV, we also give some technical hints for the evaluation of our results. In Sec. VI, we give an example for the application of our calculations. Further examples can also be found in Ref. [9]. In Sec. VII we summarize and conclude. Some special formulae are proven in an appendix.

II. DEFINITION OF A, B_0 AND C_0

We define the functions A, B_0 and C_0 to be the analytical continuation of the one, two and three fermion line integrals,

$$A(m,\mu,\ \beta,\Lambda) = \frac{16\pi^2}{\beta}\sum_n \exp(i\omega_n\eta)\int\frac{d^3p}{(2\pi)^3}\frac{1}{(i\omega_n+\mu)^2-E^2} \quad , \tag{2.1}$$

$$B_0(k,\ m_1,\mu_1,\ m_2,\mu_2,i\nu_m,\ \beta,\Lambda) = \tag{2.2}$$

$$\frac{16\pi^2}{\beta}\sum_n \exp(i\omega_n\eta)\int\frac{d^3p}{(2\pi)^3}\frac{1}{((i\omega_n+\mu_1)^2-E_1^2)}\frac{1}{((i\omega_n-i\nu_m+\mu_2)^2-E_2^2)} \quad ,$$

and

$$C_0(k,q,\delta_{\vec{k},\vec{q}},\ m_1,\mu_1,\ m_2,\mu_2,i\nu_m,\ m_3,\mu_3,i\alpha_l,\ \beta,\Lambda) = \tag{2.3}$$

$$\frac{16\pi^2}{\beta}\sum_n \exp(i\omega_n\eta)\int\frac{d^3p}{(2\pi)^3}\frac{1}{((i\omega_n+\mu_1)^2-E_1^2)}\frac{1}{((i\omega_n-i\nu_m+\mu_2)^2-E_2^2)}$$

$$\times\frac{1}{((i\omega_n-i\alpha_l+\mu_3)^2-E_3^2)} \quad ,$$

in which the complex frequencies $i\nu_m$ and $i\alpha_l$ are to be continued to their values on the real axis after the Matsubara summation on n is carried out. It is to be understood that the limit $\eta\to 0$ is to be taken after the Matsubara summation. In the above expressions, we have introduced the abbreviations

$$E = \sqrt{p^2+m^2} \qquad E_1 = \sqrt{p^2+m_1^2}$$
$$E_2 = \sqrt{(\vec{p}-\vec{k})^2+m_2^2} \qquad E_3 = \sqrt{(\vec{p}-\vec{q})^2+m_3^2} \quad , \tag{2.4}$$

where the symbols m_i and μ_i denote the masses and chemical potentials of the fermions running on the individual lines of the diagram. The m_i are to be understood to include a negative infinitesimal imaginary part, $m_i \to m_i - i\epsilon$, $\epsilon > 0$. In Eqs. (2.1) to (2.3), β is the inverse temperature, $\beta = 1/T$. The three momentum cutoff is denoted by Λ, and all integrals are considered for $|\vec{p}| \leq \Lambda$.

The complex four momenta $(i\nu_m; \vec{k})$ and $(i\alpha_l; \vec{q})$, are the four momenta that enter the loop. The frequencies $i\nu_m$ and $i\alpha_l$ are bosonic in nature,

$$\nu_m = \frac{2m\pi}{\beta} \qquad\qquad \alpha_l = \frac{2l\pi}{\beta} \quad , \tag{2.5}$$

while the frequencies $i\omega_n$ are fermionic,

$$\omega_n = \frac{(2n+1)\pi}{\beta} \quad . \tag{2.6}$$

After analytical continuation, $i\nu_m$ and $i\alpha_l$ become the zeroth components associated with the three momenta $\vec{k}$ and $\vec{q}$, i. e. $i\nu_m \to k_0$, $i\alpha_l \to q_0$. One may note at this point that, due to rotational invariance, the functions B_0 and C_0 do not fully depend on both $\vec{k}$ and $\vec{q}$; B_0 depends only on $k = |\vec{k}|$, while C_0 depends only on k, $q = |\vec{q}|$ and the enclosed angle $\delta_{\vec{k},\vec{q}}$, as is implied by the arguments specified on the left hand side of Eqs. (2.2) and (2.3).

III. THE CALCULATION OF A

The calculation of A is simple and straightforward. After evaluating the Matsubara sum by contour integration in the usual fashion [2], one has

$$\begin{aligned} A(m,\mu,\ \beta,\Lambda) &= 16\pi^2 \int \frac{d^3p}{(2\pi)^3} \left[f(E-\mu)\frac{1}{2E} - f(-E-\mu)\frac{1}{2E} \right] \\ &= 4 \int_m^{\Lambda_E} dE\, \sqrt{E^2 - m^2}\, \left(f(E-\mu) - f(-E-\mu) \right) \quad , \end{aligned} \tag{3.1}$$

where we have introduced the Fermi distribution function

$$f(x) = \frac{1}{\exp(\beta x) + 1} \tag{3.2}$$

and the energy cutoff

$$\Lambda_E = \sqrt{\Lambda^2 + m^2} \quad . \tag{3.3}$$

From Eq. (3.1), it follows that $A(m,\mu,\ \beta,\Lambda) = A(m,-\mu,\ \beta,\Lambda)$. This integral can be easily performed by numerical integration. At $T=0$, it has a closed analytical form,

$$A(m,\mu,\ \infty,\Lambda) = 2\Theta(\Lambda_E^2-\mu^2)\left(m^2\log\frac{\Lambda+\Lambda_E}{\kappa+\kappa'}+\kappa\kappa'-\Lambda\Lambda_E\right) \quad , \tag{3.4}$$

where $\kappa = \max(m,|\mu|)$ and $\kappa' = \sqrt{\kappa^2-m^2}$.

In Fig. 1, we show A/Λ^2 as a function of m/Λ, for the specific parameter choice $\mu=0$, $\beta\to\infty$ and $\Lambda = 602.3$MeV. At $m=0$, we obtain the limit $A(0,0,\ \infty,\Lambda) = -2\Lambda^2$. For larger values of m, A increases continously to zero.

IV. THE CALCULATION OF B_0

A. Matsubara Summation

The two line integral $B_0(k,\ m_1,\mu_1,\ m_2,\mu_2,i\nu_m,\ \beta,\Lambda)$ can be analyzed in a similar fashion to $A(m,\mu,\ \beta,\Lambda)$. The denominator of Eq. (2.2) has the four zeros

$$i\omega_n = -\mu_1 \pm E_1 \qquad\qquad i\omega_n = i\nu_m-\mu_2\pm E_2 \quad , \tag{4.1}$$

from which we obtain, after the summation over n, the terms

$$\begin{aligned} &B_0(k,\ m_1,\mu_1,\ m_2,\mu_2,i\nu_m,\ \beta,\Lambda)\\ &= 16\pi^2\int\frac{d^3p}{(2\pi)^3}\left[\frac{f(E_1-\mu_1)}{2E_1}\frac{1}{(-\mu_1+E_1-i\nu_m+\mu_2)^2-E_2^2}\right.\\ &\qquad -\frac{f(-E_1-\mu_1)}{2E_1}\frac{1}{(-\mu_1-E_1-i\nu_m+\mu_2)^2-E_2^2}\\ &\qquad +\frac{f(E_2-\mu_2)}{2E_2}\frac{1}{(i\nu_m-\mu_2+E_2+\mu_1)^2-E_1^2}\\ &\qquad \left.-\frac{f(-E_2-\mu_2)}{2E_2}\frac{1}{(i\nu_m-\mu_2-E_2+\mu_1)^2-E_1^2}\right] \quad . \end{aligned} \tag{4.2}$$

To eliminate the angular dependence in the arguments of the Fermi functions, we carry out the substitution

$$\vec{p}\to\vec{k}-\vec{p} \tag{4.3}$$

in the third and fourth terms of Eq. (4.2) and obtain

$$B_0(k,\ m_1,\mu_1,\ m_2,\mu_2,i\nu_m,\ \beta,\Lambda)$$
$$= 8\pi^2\int\frac{d^3p}{(2\pi)^3}\left[\frac{f(E_1-\mu_1)}{E_1}\frac{1}{\lambda^2-2\lambda E_1+2\vec{p}\vec{k}-k^2+m_1^2-m_2^2}\right.$$
$$-\frac{f(-E_1-\mu_1)}{E_1}\frac{1}{\lambda^2+2\lambda E_1+2\vec{p}\vec{k}-k^2+m_1^2-m_2^2}$$
$$+\frac{f(E_2-\mu_2)}{E_2}\frac{1}{\lambda^2+2\lambda E_2+2\vec{p}\vec{k}-k^2-m_1^2+m_2^2}$$
$$\left.-\frac{f(-E_2-\mu_2)}{E_2}\frac{1}{\lambda^2-2\lambda E_2+2\vec{p}\vec{k}-k^2-m_1^2+m_2^2}\right] \tag{4.4}$$

where the meaning of E_1 and E_2 has changed to become

$$E_1=\sqrt{p^2+m_1^2} \qquad\qquad E_2=\sqrt{p^2+m_2^2} \quad . \tag{4.5}$$

We have also introduced

$$\lambda = i\nu_m+\mu_1-\mu_2 \quad . \tag{4.6}$$

Note that λ will be *real* after the analytical continuation. One should further note that the substitution Eq. (4.3) has been performed under the assumption that $\Lambda \gg k$. Thus no corresponding shift of variables in the integration limits has been carried out at this point.

Without loss of generality, we may choose the coordinate system in such a way that $\vec{k}$ points in the z direction. Then the integration over the angle ϕ yields a factor 2π. It is now useful to consider each term occurring in Eq. (4.4) separately. These correspond to the individual poles in the original expression and have a common generic structure. One may therefore decompose B_0 into the four terms

$$B_0(k,\ m_1,\mu_1,\ m_2,\mu_2,i\nu_m,\ \beta,\Lambda) = \tilde{B}_0^+(-\lambda,k,\ m_1,m_2,\mu_1,\ \beta,\Lambda)$$
$$-\tilde{B}_0^-(\lambda,k,\ m_1,m_2,\mu_1,\ \beta,\Lambda)$$
$$+\tilde{B}_0^+(\lambda,k,\ m_2,m_1,\mu_2,\ \beta,\Lambda)$$
$$-\tilde{B}_0^-(-\lambda,k,\ m_2,m_1,\mu_2,\ \beta,\Lambda) \tag{4.7}$$

in terms of the generic function

$$\tilde{B}_0^{\pm}(\lambda, k,\ m, m', \mu,\ \beta, \Lambda) =$$

$$2\int_m^{\Lambda_E} dE\, pf(\pm E-\mu)\int_{-1}^{+1} dx \frac{1}{\lambda^2+2\lambda E+2pkx-k^2+m^2-m'^2} \tag{4.8}$$

with $p=\sqrt{E^2-m^2}$. Numerical computation of B_0 thus relies purely on the evaluation of $\tilde{B}_0^{\pm}$. The analytical structure of this integral differs for $k=0$ and $k>0$, so that these two cases must be considered separately. We examine these in the following two subsections. We may also note that the symmetry relation

$$B_0(k,\ m_1,\mu_1,\ m_2,\mu_2,-k_0,\ \beta,\Lambda) = B_0(k,\ m_2,\mu_2,\ m_1,\mu_1,k_0,\ \beta,\Lambda) \quad , \tag{4.9}$$

can be inferred from Eq. (4.4).

B. Calculation of $\tilde{B}_0^{\pm}$ for $k=0$

For $k=0$, the integral over x is trivial, and we obtain

$$\tilde{B}_0^{\pm} = 4\int_m^{\Lambda_E} dE \frac{\sqrt{E^2-m^2}f(\pm E-\mu)}{\lambda^2+2\lambda E+m^2-m'^2} \quad . \tag{4.10}$$

(Here and in the following, we drop the arguments of $\tilde{B}_0^{\pm}$ for convenience.) The analytical continuation of the Matsubara frequency $i\nu_m$ to real values leads to poles in the integrand at $E=E_0$, where

$$E_0 = -\frac{\lambda^2+m^2-m'^2}{2\lambda} \quad , \tag{4.11}$$

if $m \le E_0 \le \Lambda_E$. To calculate the integral in this case, we recall that the particle masses are complex, $m^2 \to m^2 - i\epsilon$, $m'^2 \to m'^2 - i\epsilon$, and apply the formula

$$\lim_{\epsilon\to 0}\frac{1}{x-i\epsilon} = \mathcal{P}\frac{1}{x} + i\pi\delta(x) \quad , \tag{4.12}$$

where $\mathcal{P}$ denotes the Cauchy principal value. From this, we find

$$\tilde{B}_0^{\pm} = \lim_{\epsilon\to 0} 4\int_m^{\Lambda_E} dE \frac{pf(\pm E-\mu)}{\lambda^2+2\lambda E+m^2-m'^2-i\epsilon\,\mathrm{sgn}(\lambda)}$$

$$= \lim_{\epsilon\to 0}\frac{2}{\lambda}\int_m^{\Lambda_E} dE \frac{pf(\pm E-\mu)}{E+\frac{\lambda^2+m^2-m'^2}{2\lambda}-i\epsilon}$$

$$= 4\,\mathcal{P}\!\!\int_m^{\Lambda_E} dE \frac{pf(\pm E-\mu)}{\lambda^2+2\lambda E+m^2-m'^2}$$

$$+\, i\frac{2\pi}{\lambda}p_0 f(\pm E_0-\mu)\Theta((\Lambda_E-E_0)(E_0-m)) \quad , \tag{4.13}$$

where $p_0=\sqrt{E_0^2-m^2}$ and the Θ function ensures that the imaginary part appears only if E_0 lies within the integration interval $[m,\Lambda_E]$.

Figure 2 shows B_0 as a function of k_0/Λ for the parameter set $k=0$, $\mu_1=\mu_2=0$, $m_1=m_2=m=367.7$MeV, $\beta\to\infty$, $\Lambda=602.3$MeV. One recognizes that at small k_0, B_0 is a real function, whereas for $k_0\geq 2m$ an imaginary part emerges. At $k_0>2\Lambda_E$, the imaginary part vanishes due to the presence of the cutoff.

C. Calculation of $\tilde{B}_0^{\pm}$ for $k>0$

For $k>0$, the generic function $\tilde{B}_0^{\pm}$ has the form

$$\tilde{B}_0^{\pm}(\lambda,k,\ m,m',\mu,\ \beta,\Lambda)=2\lim_{\epsilon\to 0}\int_m^{\Lambda_E} dE\, pf(\pm E-\mu)$$

$$\times\int_{-1}^{+1} dx\frac{1}{\lambda^2+2\lambda E+2pkx-k^2+m^2-m'^2-i\epsilon\mathrm{sgn}(\lambda)} \quad , \tag{4.14}$$

where we have already indicated that the infinitesimal imaginary part in the integrand is determined by the sign of λ. In the limit $\epsilon\to 0$, the integrand contains a singularity at

$$x=-\frac{1}{2pk}(\lambda^2+2\lambda E-k^2+m^2-m'^2) \tag{4.15}$$

if

$$|\lambda^2+2\lambda E-k^2+m^2-m'^2|<2pk \quad . \tag{4.16}$$

On performing the x integration and applying Eq. (4.12), one obtains

$$\tilde{B}_0^{\pm}=\lim_{\epsilon\to 0}\frac{1}{k}\int_m^{\Lambda_E} dE\, f(\pm E-\mu)\int_{-1}^{+1} dx\frac{1}{x+\frac{\lambda^2+2\lambda E-k^2+m^2-m'^2}{2pk}-i\epsilon\mathrm{sgn}(\lambda)} \tag{4.17}$$

$$=\frac{1}{k}\int_m^{\Lambda_E} dE\, f(\pm E-\mu)\log\left|\frac{(\lambda+E)^2-(p-k)^2-m'^2}{(\lambda+E)^2-(p+k)^2-m'^2}\right|$$

$$+\, i\frac{\pi\mathrm{sgn}(\lambda)}{k}\int_m^{\Lambda_E} dE\,\Theta(2pk-|\lambda^2+2\lambda E-k^2+m^2-m'^2|)f(\pm E-\mu) \quad .$$

The integral for the imaginary part can be done analytically:

$$\int dE \frac{1}{\exp(\pm\beta E - \beta\mu) + 1} = E \mp \frac{1}{\beta} \log(1 + \exp(\pm\beta E - \beta\mu)) \quad . \tag{4.18}$$

The integrand for the real part in Eq. (4.17) contains logarithmic poles at the energies $E = E_{1/2}$ which are defined by

$$|\lambda^2 + 2\lambda E_{1/2} - k^2 + m^2 - m'^2| = 2k\sqrt{E_{1/2}^2 - m^2} \quad . \tag{4.19}$$

These energies also constitute the limits of integration for the imaginary part. Squaring Eq. (4.19) leads to a quadratic equation for $E_{1/2}$, which has the solutions

$$E_{1/2} = \frac{\lambda(\lambda^2 - k^2 + m^2 - m'^2)}{2(k^2 - \lambda^2)} \pm k \frac{\sqrt{(\lambda^2 - k^2 + m^2 - m'^2)^2 - 4m^2(k^2 - \lambda^2)}}{2(k^2 - \lambda^2)} \quad . \tag{4.20}$$

There are three possible cases for Eq. (4.20):

1. No solutions exist for $E_{1/2}$, or both solutions lie outside the integration interval $[m, \Lambda_E]$. Since there is at least one point ($E = m$), where the Θ function in Eq. (4.17) vanishes, we conclude that in this case the imaginary part disappears.

2. One solution E_1 lies inside the integration interval, the other one, E_2 lies outside. It follows that the integral for the imaginary part has to be performed from E_1 to Λ_E in this case.

3. The two solutions $E_1 < E_2$ lie within the integration interval. In this case, the integral for the imaginary part has to be performed from E_1 to E_2.

Since we will deal with integration limits of this type very often, we will adopt the convention that the solutions $E_{1/2}$ of Eq. (4.20) are tailored as described above. Eq. (4.17) then takes the form

$$\tilde{B}_0^\pm = \frac{1}{k} \int_m^{\Lambda_E} dE\, f(\pm E - \mu) \log \left| \frac{(\lambda + E)^2 - (p - k)^2 - m'^2}{(\lambda + E)^2 - (p + k)^2 - m'^2} \right| \tag{4.21}$$

$$+ i \frac{\pi \mathrm{sgn}(\lambda)}{k} \int_{E_1}^{E_2} dE\, f(\pm E - \mu) \quad .$$

Together with Eq. (4.20), Eq. (4.21) provides a sufficient basis for the numerical calculation of $\tilde{B}_0^{\pm}$. It can be shown that in the limit $k \to 0$, Eq. (4.21) continously approaches Eq. (4.13). Thus the generic function $\tilde{B}_0^{\pm}$ is determined, and consequently B_0 via Eq. (4.7).

Figure 3 shows B_0 for the same parameter set as was used in Fig. 2, but with $k = 100$MeV. For $k_0 < k$, Eq. (4.20) has one solution, from which one obtains an imaginary part. For $k_0 > 2m$, Eq. (4.20) has two solutions, which become cut off at high k_0, so that the imaginary part continously goes to zero. Figure 4 shows another special case. Here we plot B_0 as a function of k for $k_0 = 0$. In this case, we have no imaginary part at all [7].

D. Programming Considerations

At this point, we deviate from the analytical calculation to discuss some of the basic aspects of programming such integrals. Comments made here are relevant for the B_0 and C_0 integrals, the first of which is the least complicated integral containing multiple singularities. We discuss this now, bearing in mind that similar techniques are required for the more complex C_0 integral.

To calculate B_0 for the most general case via the decomposition (4.7), one has to take several things into consideration. First, the type of singularity determines the type of integration routine that should be utilized. One requires three such routines: (i) one routine which integrates smooth functions, (ii) one which integrates functions containing logarithmic or other integrable singularities, and (iii) one which computes the Cauchy principal value of an integral containing a $1/x$ singularity. Examples for this can be found in the literature [11,12].

On entry, the program has to decide, which of the forms (4.13), pertinent to the case $k = 0$, or (4.21), pertinent for $k > 0$, to use. It is necessary to handle the two cases separately, since (i) Eq. (4.13) and Eq. (4.21) have a different analytical structure, and (ii) Eq. (4.21) contains a factor $1/k$. In actual practice, the case $k = 0$ is needed very often. After that, the position of singularities can be computed according to Eq. (4.11) or Eq. (4.20), respectively. The appropriate integration routine has to be chosen, depending on the result of this computation.

Note that although the decomposition as given Eq. (4.7) is the most general

one possible, it does not always provide the optimal way of calculating B_0. If, for example, one knows a priori that an actual calculation will be confined to $m_1 = m_2$, $\mu_1 = \mu_2$ and $k = 0$, the four terms of Eq. (4.7) can be combined, yielding the simplified expression

$$B_0(0,\ m,\mu,\ m,\mu,k_0,\ \beta,\Lambda) = 8\,\mathcal{P}\!\int_m^{\Lambda_E} dE(f(E-\mu) - f(-E-\mu))\frac{p}{k_0^2 - 4E^2} \tag{4.22}$$

$$-i\pi\left(f\left(\frac{k_0}{2}-\mu\right) - f\left(-\frac{k_0}{2}-\mu\right)\right)\frac{\sqrt{k_0^2-4m^2}}{k_0}\Theta((2\Lambda_E - k_0)(k_0-2m)) \quad .$$

The result of this equation stays finite in the limit $k_0 \to 0$. The functions $\tilde{B}_0^{\pm}$, however, individually *diverge* in this limit, so that a computation of B_0 by decomposition suffers from cancellations at small k_0. Note that similar simplifications follow for C_0.

V. THE CALCULATION OF C_0

A. Matsubara Summation

In this section, we examine the analytical structure of the three fermion line integral C_0, and decompose it into a sum involving a single generic function.

The three fermion line function given in Eq. (2.3) has poles at

$$i\omega_n = -\mu_1 \pm E_1 \tag{5.1a}$$

$$i\omega_n = i\nu_m - \mu_2 \pm E_2 \tag{5.1b}$$

$$i\omega_n = i\alpha_l - \mu_3 \pm E_3 \tag{5.1c}$$

with E_1, E_2 and E_3 as specified in Eq. (2.4). In analogy to Eq. (4.6), we define the complex variables

$$\lambda_1 = i\nu_m + \mu_1 - \mu_2 \tag{5.2a}$$

$$\lambda_2 = i\alpha_l + \mu_1 - \mu_3 \tag{5.2b}$$

$$\lambda_3 = i\nu_m - i\alpha_l - \mu_2 + \mu_3 = \lambda_1 - \lambda_2 \quad . \tag{5.2c}$$

We again note that these parameters become real numbers after the analytical continuation is performed.

After summing the Matsubara frequencies, one obtains a sum over six terms that arise from each of the poles,

$$
\begin{aligned}
&C_0(k, q, \delta_{\vec{k},\vec{q}},\ m_1, \mu_1,\ m_2, \mu_2, i\nu_m,\ m_3, \mu_3, i\alpha_l,\ \beta, \Lambda) \\
&= 8\pi^2 \int \frac{d^3p}{(2\pi)^3} \left[\frac{f(E_1 - \mu_1)}{E_1} \frac{1}{[(\lambda_1 - E_1)^2 - E_2^2][(\lambda_2 - E_1)^2 - E_3^2]} \right. \\
&\qquad + (E_1 \to -E_1) \\
&\qquad + \frac{f(E_2 - \mu_2)}{E_2} \frac{1}{[(\lambda_1 + E_2)^2 - E_1^2][(\lambda_3 + E_2)^2 - E_3^2]} \\
&\qquad + (E_2 \to -E_2) \\
&\qquad + \frac{f(E_3 - \mu_3)}{E_3} \frac{1}{[(\lambda_2 + E_3)^2 - E_1^2][(\lambda_3 - E_3)^2 - E_2^2]} \\
&\qquad \left. + (E_3 \to -E_3) \right] \quad .
\end{aligned}
\tag{5.3}
$$

It is useful to make the substitution $\vec{p} \to \vec{k} - \vec{p}$ in the third and fourth terms, $\vec{p} \to \vec{q} - \vec{p}$ in the fifth and sixth terms, in order to get rid of the angular dependence in the arguments of the Fermi functions. Again this is performed under the assumption that $\Lambda \gg k, q$. After this, we obtain a decomposition of C_0 of the form

$$
\begin{aligned}
&C_0(k, q, \delta_{\vec{k},\vec{q}},\ m_1, \mu_1,\ m_2, \mu_2, i\nu_m,\ m_3, \mu_3, i\alpha_l,\ \beta, \Lambda) \\
&\quad = \tilde{C}_0^+(-\lambda_1, -\lambda_2,\ k, q, \delta_{\vec{k},\vec{q}},\ m_1, m_2, m_3, \mu_1,\ \beta, \Lambda) \\
&\quad - \tilde{C}_0^-(\lambda_1, \lambda_2,\ k, q, \delta_{\vec{k},\vec{q}},\ m_1, m_2, m_3, \mu_1,\ \beta, \Lambda) \\
&\quad + \tilde{C}_0^+(\lambda_1, \lambda_3,\ k, |\vec{k} - \vec{q}|, \delta_{\vec{k},\vec{k}-\vec{q}},\ m_2, m_1, m_3, \mu_2,\ \beta, \Lambda) \\
&\quad - \tilde{C}_0^-(-\lambda_1, -\lambda_3,\ k, |\vec{k} - \vec{q}|, \delta_{\vec{k},\vec{k}-\vec{q}},\ m_2, m_1, m_3, \mu_2,\ \beta, \Lambda)
\end{aligned}
$$

$$+\tilde{C}_0^{+}(\lambda_2,-\lambda_3,\ q,|\vec{q}-\vec{k}|,\delta_{\vec{q},\vec{q}-\vec{k}},\ m_3,m_1,m_2,\mu_3,\ \beta,\Lambda)$$

$$-\tilde{C}_0^{-}(-\lambda_2,\lambda_3,\ q,|\vec{q}-\vec{k}|,\delta_{\vec{q},\vec{q}-\vec{k}},\ m_3,m_1,m_2,\mu_3,\ \beta,\Lambda) \quad , \tag{5.4}$$

where we have introduced the three fermion line generic function

$$\tilde{C}_0^{\pm}(\lambda_1,\lambda_2,\ k,q,\delta,\ m,m_1,m_2,\mu,\ \beta,\Lambda) = 8\pi^2 \tag{5.5}$$

$$\times \quad \int \frac{d^3p}{(2\pi)^3}\frac{f(\pm E-\mu)}{E}\frac{1}{[(\lambda_1+E)^2-(\vec{p}-\vec{k})^2-m_1^2][(\lambda_2+E)^2-(\vec{p}-\vec{q})^2-m_2^2]}$$

with $E=\sqrt{p^2+m^2}$. In analogy to Eq. (4.9), one can derive a symmetry relation for the function C_0, as being

$$C_0(k,q,\delta_{\vec{k},\vec{q}},\ m_1,\mu_1,\ m_2,\mu_2,i\nu_m,\ m_3,\mu_3,i\alpha_l,\ \beta,\Lambda) \tag{5.6}$$

$$= C_0(k,|\vec{k}-\vec{q}|,\delta_{\vec{k},\vec{k}-\vec{q}},\ m_2,\mu_2,\ m_1,\mu_1,-i\nu_m,\ m_3,\mu_3,i\alpha_l-i\nu_m,\ \beta,\Lambda) \quad .$$

As was the case for the two line generic function $\tilde{B}_0^{\pm}$, $\tilde{C}_0^{\pm}$ is calculated for several special cases. We list these in order of increasing difficulty:

1. The case $k=q=0$.
2. The case $k=0$, $q>0$.
3. The case $k>0$, $q>0$, $\vec{p}$ and $\vec{k}$ collinear.
4. The case $k>0$, $q>0$, $\vec{p}$ and $\vec{k}$ not collinear – general case.

In the following sections, we will derive results for each of these special cases.

B. Calculation of $\tilde{C}_0^{\pm}$ for $k=q=0$

In this case, Eq. (5.5) reduces to

$$\tilde{C}_0^{\pm} = 8\pi^2\int\frac{d^3p}{(2\pi)^3}\frac{f(\pm E-\mu)}{E}\frac{1}{[(\lambda_1+E)^2-p^2-m_1^2][(\lambda_2+E)^2-p^2-m_2^2]}$$

$$= \frac{1}{\lambda_1\lambda_2}\lim_{\epsilon\to 0}\int_m^{\Lambda_E} dE\frac{pf(\pm E-\mu)}{(E-E_1-i\epsilon)(E-E_2-i\epsilon)} \quad , \tag{5.7}$$

where we have again omitted the arguments of $\tilde{C}_0^{\pm}$ for convenience and have included a small imaginary part in the denominators to do the analytical continuation. The constants E_1 and E_2 are defined by

$$E_1 = -\frac{\lambda_1^2 + m^2 - m_1^2}{2\lambda_1} \tag{5.8a}$$

$$E_2 = -\frac{\lambda_2^2 + m^2 - m_2^2}{2\lambda_2} \quad . \tag{5.8b}$$

If $m \leq E_1 = E_2 \leq \Lambda_E$, the integral in Eq. (5.7) diverges. We suppose therefore that $E_1 \neq E_2$, if one of them lies inside the integration interval. After taking the limit $\epsilon \to 0$, we obtain

$$\begin{aligned}\tilde{C}_0^{\pm} = 4\,\mathcal{P}\!\!\int_m^{\Lambda_E} dE \frac{pf(\pm E - \mu)}{[(\lambda_1 + E)^2 - p^2 - m_1^2][(\lambda_2 + E)^2 - p^2 - m_2^2]} \\ + i\frac{\pi}{\lambda_1\lambda_2(E_1 - E_2)}\Big(p_1 f(\pm E_1 - \mu)\Theta((\Lambda_E - E_1)(E_1 - m)) \\ -p_2 f(\pm E_2 - \mu)\Theta((\Lambda_E - E_2)(E_2 - m))\Big) \quad .\end{aligned} \tag{5.9}$$

The Θ functions here guarantee that the imaginary part occurs only if the singularities appear inside the integration interval. As was detailed in Sec. IV D, the numerical evaluation of the integral in Eq. (5.9) has to proceed differently if E_1 or E_2 lie within the interval $[m, \Lambda_E]$ (Cauchy integration), or if they lie outside of $[m, \Lambda_E]$ (integration of a smooth function).

Figure 5 shows $m^2 C_0$ for $k = q = 0$ as a function of $k_0/\Lambda = 2q_0/\Lambda$ for a similar parameter set as was used in Fig. 2. Again we obtain an imaginary part for $k_0 > 2m$.

C. Calculation of $\tilde{C}_0^{\pm}$ for $k = 0$, $q > 0$

Without loss of generality, we assume that $\vec{q}$ points in the z direction. With this assumption, the scalar product $\vec{p}\vec{q}$ occurring in Eq. (5.5) simplifies to $pq\cos\theta$. The integration over ϕ becomes trivial and one obtains

$$\begin{aligned}\tilde{C}_0^{\pm} = \frac{1}{\pi}\lim_{\epsilon\to 0}\int d^3p \frac{f(\pm E - \mu)}{E}\frac{1}{\lambda_1^2 + 2\lambda_1 E + m^2 - m_1^2 - i\epsilon\mathrm{sgn}(\lambda_1)} \\ \times\frac{1}{\lambda_2^2 + 2\lambda_2 E + 2pq\cos\theta - q^2 + m^2 - m_2^2 - i\epsilon\mathrm{sgn}(\lambda_2)}\end{aligned}$$

$$= \frac{1}{2\lambda_1\lambda_2} \lim_{\epsilon\to 0} \int_m^{\Lambda_E} dE \frac{pf(\pm E-\mu)}{E-E_1-i\epsilon} \int_{-1}^{+1} dx \frac{1}{E-E_2+\frac{pq}{\lambda_2}x-i\epsilon} \quad , \tag{5.10}$$

where now

$$E_2 = -\frac{\lambda_2^2 - q^2 + m^2 - m_2^2}{2\lambda_2} \tag{5.11}$$

and E_1 is as defined in Eq. (5.8a). The angular integral is singular if $|\lambda_2(E-E_2)| < pq$. This equation has exactly the same structure as Eq. (4.16), so Eq. (4.20) and the following discussion can be applied here too. We label the endpoints of the singular interval E_{21} and E_{22}. The angular integral becomes

$$\begin{aligned}
\lim_{\epsilon\to 0} \int_{-1}^{+1} dx \frac{1}{E-E_2+\frac{pq}{\lambda_2}x-i\epsilon} &= \frac{\lambda_2}{pq} \lim_{\epsilon\to 0} \int_{-1}^{+1} dx \frac{1}{\frac{\lambda_2(E-E_2)}{pq}+x-i\epsilon\,\mathrm{sgn}(\lambda_2)} \\
&= \frac{\lambda_2}{pq} \log\left|\frac{\frac{\lambda_2(E-E_2)}{pq}+1}{\frac{\lambda_2(E-E_2)}{pq}-1}\right| + i\frac{\pi\lambda_2\mathrm{sgn}(\lambda_2)}{pq}\Theta\left(1-\left|\frac{\lambda_2(E-E_2)}{pq}\right|\right) \\
&= \frac{\lambda_2}{pq} \log\left|\frac{(\lambda_2+E)^2-(p-q)^2-m_2^2}{(\lambda_2+E)^2-(p+q)^2-m_2^2}\right| \\
&\qquad + i\frac{\pi\lambda_2\mathrm{sgn}(\lambda_2)}{pq}\Theta((E_{22}-E)(E-E_{21})) \quad .
\end{aligned} \tag{5.12}$$

For $\tilde{C}_0^{\pm}$ we have accordingly

$$\begin{aligned}
\tilde{C}_0^{\pm} &= \frac{1}{2\lambda_1 q} \lim_{\epsilon\to 0} \int_m^{\Lambda_E} dE \frac{f(\pm E-\mu)}{E-E_1-i\epsilon} \log\left|\frac{(\lambda_2+E)^2-(p-q)^2-m_2^2}{(\lambda_2+E)^2-(p+q)^2-m_2^2}\right| \\
&+ i\frac{\pi\mathrm{sgn}(\lambda_2)}{2\lambda_1 q} \lim_{\epsilon\to 0} \int_{E_{21}}^{E_{22}} dE \frac{f(\pm E-\mu)}{E-E_1-i\epsilon} \quad .
\end{aligned} \tag{5.13}$$

In this expression, we again have to take the limit $\epsilon \to 0$. The final expression we find is

$$\begin{aligned}
\tilde{C}_0^{\pm} &= \frac{1}{2\lambda_1 q} \mathcal{P}\!\!\int_m^{\Lambda_E} dE \frac{f(\pm E-\mu)}{E-E_1} \log\left|\frac{(\lambda_2+E)^2-(p-q)^2-m_2^2}{(\lambda_2+E)^2-(p+q)^2-m_2^2}\right| \\
&- \frac{\pi^2\mathrm{sgn}(\lambda_2)}{2\lambda_1 q} f(\pm E_1-\mu)\Theta((E_{22}-E_1)(E_1-E_{21})) \\
&+ i\frac{\pi f(\pm E_1-\mu)}{2\lambda_1 q} \log\left|\frac{(\lambda_2+E_1)^2-(p_1-q)^2-m_2^2}{(\lambda_2+E_1)^2-(p_1+q)^2-m_2^2}\right| \Theta((\Lambda_E-E_1)(E_1-m)) \\
&+ i\frac{\pi\mathrm{sgn}(\lambda_2)}{2\lambda_1 q} \mathcal{P}\!\!\int_{E_{21}}^{E_{22}} dE \frac{f(\pm E-\mu)}{E-E_1} \quad .
\end{aligned} \tag{5.14}$$

A numerical evaluation of Eq. (5.14) has to take into account Cauchy singularities at $E = E_1$ and (integrable) logarithmic singularities at $E = E_{21}$ and $E = E_{22}$.

D. Calculation of $\tilde{C}_0^{\pm}$ for $k > 0$, $q > 0$, $\vec{q}$ and $\vec{k}$ collinear

Again, we assume without loss of generality that $\vec{q}$ and $\vec{k}$ have the form

$$\vec{q} = \begin{pmatrix} 0 \\ 0 \\ q \end{pmatrix} \qquad \vec{k} = \begin{pmatrix} 0 \\ 0 \\ \eta k \end{pmatrix} \tag{5.15}$$

where $\eta = \pm 1$. With this assumption, we have $\vec{p}\vec{q} = pq\cos\theta$ and $\vec{p}\vec{k} = \eta pk\cos\theta$ in Eq. (5.5). The ϕ integration is again trivial, and one obtains

$$\begin{aligned}
\tilde{C}_0^{\pm} &= \frac{1}{\pi}\lim_{\epsilon\to 0}\int d^3p \frac{f(\pm E-\mu)}{E}\frac{1}{\lambda_1^2 + 2\lambda_1 E + 2\eta pk\cos\theta - k^2 + m^2 - m_1^2 - i\epsilon\,\mathrm{sgn}(\lambda_1)} \\
&\quad\times\frac{1}{\lambda_2^2 + 2\lambda_2 E + 2pq\cos\theta - q^2 + m^2 - m_2^2 - i\epsilon\,\mathrm{sgn}(\lambda_2)} \\
&= \frac{1}{2\lambda_1\lambda_2}\lim_{\epsilon\to 0}\int_m^{\Lambda_E} dE\, p f(\pm E-\mu) \\
&\qquad\times\int_{-1}^{+1} dx \frac{1}{E - E_1 + \frac{\eta pk}{\lambda_1}x - i\epsilon}\,\frac{1}{E - E_2 + \frac{pq}{\lambda_2}x - i\epsilon}
\end{aligned} \tag{5.16}$$

with

$$E_1 = -\frac{\lambda_1^2 - k^2 + m^2 - m_1^2}{2\lambda_1} \tag{5.17}$$

and E_2 from Eq. (5.11). The integral over x becomes (cf. [13], integral No. 12.8)

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_{-1}^{+1} dx \frac{1}{E - E_1 + \frac{\eta pk}{\lambda_1}x - i\epsilon}\,\frac{1}{E - E_2 + \frac{pq}{\lambda_2}x - i\epsilon} \\
&\quad= \frac{\lambda_1\lambda_2}{pq\lambda_1(E-E_1) - \eta pk\lambda_2(E-E_2)} \\
&\qquad\times\log\left|\frac{(\lambda_2(E-E_2)+pq)(\lambda_1(E-E_1)-\eta pk)}{(\lambda_2(E-E_2)-pq)(\lambda_1(E-E_1)+\eta pk)}\right|
\end{aligned}$$

$$+i\frac{\pi\lambda_1\lambda_2\eta\mathrm{sgn}(\lambda_1)\Theta\left(1-\left|\frac{\lambda_1(E-E_1)}{pk}\right|\right)}{\eta pk\lambda_2(E-E_2)-pq\lambda_1(E-E_1)}$$

$$+i\frac{\pi\lambda_1\lambda_2\mathrm{sgn}(\lambda_2)\Theta\left(1-\left|\frac{\lambda_2(E-E_2)}{pq}\right|\right)}{pq\lambda_1(E-E_1)-\eta pk\lambda_2(E-E_2)} \quad . \tag{5.18}$$

In what follows, we label the energies for which

$$|\lambda_1(E-E_1)| = pk \tag{5.19}$$

is fulfilled $E_{1\,1/2}$, the energies for which

$$|\lambda_2(E-E_2)| = pq \quad . \tag{5.20}$$

is fulfilled $E_{2\,1/2}$. These can be calculated explicitly via Eq. (4.20). Note that the discussion following Eq. (4.20) applies here too.

The second and third terms of Eq. (5.18) still contain a singularity at $E = E_0$, where

$$E_0 = \frac{2}{\zeta}(q\lambda_1 E_1 - \eta k\lambda_2 E_2) \tag{5.21}$$

with

$$\zeta = 2(q\lambda_1 - \eta k\lambda_2) \quad . \tag{5.22}$$

This is, however, not true for the first term, since the logarithmic factor goes to zero at $E = E_0$, making the integrand continous here. So we again add a small imaginary part in the denominators where appropriate and obtain

$$\begin{aligned}
\tilde{C}_0^{\pm} &= \frac{1}{\zeta}\int_m^{\Lambda_E} dE\frac{f(\pm E-\mu)}{E-E_0}\log\left|\frac{(\lambda_2(E-E_2)+pq)(\lambda_1(E-E_1)-\eta pk)}{(\lambda_2(E-E_2)-pq)(\lambda_1(E-E_1)+\eta pk)}\right| \\
&- i\frac{\pi\eta\mathrm{sgn}(\lambda_1)}{\zeta}\lim_{\epsilon\to 0}\int_{E_{11}}^{E_{12}} dE\frac{f(\pm E-\mu)}{E-E_0-i\epsilon} + i\frac{\pi\mathrm{sgn}(\lambda_2)}{\zeta}\lim_{\epsilon\to 0}\int_{E_{21}}^{E_{22}} dE\frac{f(\pm E-\mu)}{E-E_0-i\epsilon} \\
&= \frac{1}{\zeta}\int_m^{\Lambda_E} dE\frac{f(\pm E-\mu)}{E-E_0}\log\left|\frac{(\lambda_2(E-E_2)+pq)(\lambda_1(E-E_1)-\eta pk)}{(\lambda_2(E-E_2)-pq)(\lambda_1(E-E_1)+\eta pk)}\right| \\
&+ \frac{\pi^2\eta\mathrm{sgn}(\lambda_1)}{\zeta}f(\pm E_0-\mu)\Theta((E_{12}-E_0)(E_0-E_{11}))
\end{aligned}$$

$$- \frac{\pi^2 \mathrm{sgn}(\lambda_2)}{\zeta} f(\pm E_0 - \mu)\Theta((E_{22} - E_0)(E_0 - E_{21}))$$

$$- i\frac{\pi\eta \mathrm{sgn}(\lambda_1)}{\zeta} \mathcal{P}\!\!\int_{E_{11}}^{E_{12}} dE \frac{f(\pm E - \mu)}{E - E_0} + i\frac{\pi \mathrm{sgn}(\lambda_2)}{\zeta} \mathcal{P}\!\!\int_{E_{21}}^{E_{22}} dE \frac{f(\pm E - \mu)}{E - E_0} \quad . \tag{5.23}$$

For the numerical evaluation of the real part, one has to take into account the logarithmic poles at $E = E_{1\,1/2}$ and $E = E_{2\,1/2}$. For the imaginary part, the Cauchy singularity at $E = E_0$ must be integrated.

In Fig. 6, we show $m^2 C_0$ as a function of k_0/Λ for $q = q_0 = k_0/2$ and all other parameters as in Fig. 5. This special case has been considered previously in Ref. [8]. As can be seen from Eq. (5.4), the computation of C_0 requires a computation of $\tilde{C}_0^{\pm}$ for $k = 0$, $q > 0$, as well as for $\vec{k}$, $\vec{q}$ collinear.

E. Calculation of $\tilde{C}_0^{\pm}$ for the General Case

In this subsection, we handle the general case of arbitrary momenta in the evaluation of $\tilde{C}_0^{\pm}$ from Eq. (5.5). We choose the coordinate system in such a way that

$$\vec{q} = \begin{pmatrix} 0 \\ 0 \\ q \end{pmatrix} \qquad \vec{k} = \begin{pmatrix} 0 \\ k\sin\delta \\ k\cos\delta \end{pmatrix} \quad . \tag{5.24}$$

From this choice we obtain

$$\vec{p}\vec{k} = pk(\cos\theta\cos\delta + \sin\theta\sin\delta\cos\phi) \tag{5.25a}$$

$$\vec{p}\vec{q} = pq\cos\theta \quad . \tag{5.25b}$$

It is useful to introduce the abbreviations

$$c_1 = \frac{\lambda_1^2 + 2\lambda_1 E + m^2 - m_1^2 - k^2}{2pk} \tag{5.26a}$$

$$c_2 = \frac{\lambda_2^2 + 2\lambda_2 E + m^2 - m_2^2 - q^2}{2pq} \quad . \tag{5.26b}$$

In terms of these variables, $\tilde{C}_0^{\pm}$ takes the form

$$\tilde{C}_0^{\pm} = \frac{1}{4\pi kq} \lim_{\epsilon \to 0} \int_m^{\Lambda_E} dE \frac{f(\pm E - \mu)}{p} \int_0^{\pi} d\theta \frac{\sin\theta}{c_2 + \cos\theta - i\epsilon \mathrm{sgn}(\lambda_2)}$$

$$\times \int_0^{2\pi} d\phi \frac{1}{c_1 + \cos\theta\cos\delta + \sin\theta\sin\delta\cos\phi - i\epsilon\mathrm{sgn}(\lambda_1)} \quad . \tag{5.27}$$

The additional difficulty introduced by the generality of handling arbitrary momenta is evident in the ϕ integration. As will be seen in the following, this can however still be dealt with analytically.

To perform the ϕ integration, we use the formula

$$\lim_{\epsilon \to 0} \int_0^{2\pi} \frac{d\phi}{a + b\cos\phi - i\epsilon} = \frac{2\pi}{\sqrt{|a^2 - b^2|}} \Big(\Theta(a^2 - b^2)\mathrm{sgn}(a) + i\Theta(b^2 - a^2)\Big) \quad , \tag{5.28}$$

which we prove in Appendix A. With this formula, one immediately obtains the result

$$\tilde{C}_0^{\pm} = \frac{1}{2kq} \int_m^{\Lambda_E} dE \frac{f(\pm E - \mu)}{p} \int_{-1}^{+1} dx \frac{\Theta(\Delta_1)\mathrm{sgn}(c_1 + x\cos\delta) + i\Theta(-\Delta_1)\mathrm{sgn}(\lambda_1)}{\sqrt{|\Delta_1|}(x + c_2 - i\epsilon\mathrm{sgn}(\lambda_2))} \tag{5.29}$$

where

$$\Delta_1 = (c_1 + x\cos\delta)^2 - \sin^2\delta(1 - x^2) \quad . \tag{5.30}$$

In the following subsections, we examine the remaining angular integrals that are required.

1. The θ Integration for $\Delta_1 < 0$

In the next step, we calculate the integral

$$\int_{-1}^{+1} dx \frac{\Theta(-\Delta_1)\mathrm{sgn}(\lambda_1)}{\sqrt{|\Delta_1|}(x + c_2 - i\epsilon\mathrm{sgn}(\lambda_2))} \quad . \tag{5.31}$$

We first note that for $x = \pm 1$, Δ_1 is always nonnegative. Δ_1 has zeros at

$$x = x_1 = -c_1\cos\delta - \sqrt{1 - c_1^2}\sin\delta \tag{5.32a}$$

and

$$x = x_2 = -c_1 \cos\delta + \sqrt{1-c_1^2}\sin\delta \tag{5.32b}$$

with $-1 \leq x_1 \leq x_2 \leq +1$. Consequently two conditions must be fulfilled in order to obtain $\Delta_1 \leq 0$: (i) $|c_1| < 1$ and (ii) $x_1 \leq x \leq x_2$. From this it follows that we can apply the formula

$$\lim_{\epsilon\to 0}\int_a^b \frac{dx}{\sqrt{(b-x)(x-a)}(x+c-i\epsilon)} = \frac{\pi}{\sqrt{|\Delta|}}\left(\overline{\mathrm{sgn}(x+c)}\Theta(\Delta) + i\Theta(-\Delta)\right) \quad , \tag{5.33}$$

where $\Delta = (a+c)(b+c)$, which is proven in Appendix B. Note that for $\Delta > 0$ $\mathrm{sgn}(x+c)$ is a constant in the interval $[a, b]$, which is expressed by the symbol $\overline{\mathrm{sgn}(x+c)}$.

Using Eq. (5.33), one obtains

$$\int_{-1}^{+1} \frac{\Theta(-\Delta_1)\mathrm{sgn}(\lambda_1)}{\sqrt{|\Delta_1|}(x+c_2-i\epsilon\,\mathrm{sgn}(\lambda_2))}dx \tag{5.34}$$

$$= \Theta(1-c_1^2)\frac{\pi\,\mathrm{sgn}(\lambda_1)}{\sqrt{|\Delta_0|}}\left(\mathrm{sgn}(c_2 - c_1\cos\delta)\Theta(\Delta_0) + i\,\mathrm{sgn}(\lambda_2)\Theta(-\Delta_0)\right) \quad ,$$

where the factor $\overline{\mathrm{sgn}(x+c_2)}$ has been taken at $x = (x_1+x_2)/2 = -c_1\cos\delta$. Δ_0 can be obtained from Δ_1 by substituting $-c_2$ for x:

$$\Delta_0 = c_1^2 + c_2^2 - 2c_1c_2\cos\delta - \sin^2\delta \tag{5.35}$$

2. The θ Integration for $\Delta_1 > 0$

It remains to compute the integral

$$\lim_{\epsilon\to 0}\int_{-1}^{+1} \frac{\Theta(\Delta_1)\mathrm{sgn}(c_1 + x\cos\delta)}{\sqrt{\Delta_1}(x+c_2-i\epsilon\,\mathrm{sgn}(\lambda_2))}dx = \mathcal{P}\!\int_{-1}^{+1} \frac{\Theta(\Delta_1)\mathrm{sgn}(c_1+x\cos\delta)}{\sqrt{\Delta_1}(x+c_2)}dx$$

$$+i\frac{\pi\,\mathrm{sgn}(\lambda_2)\mathrm{sgn}(c_1 - c_2\cos\delta)}{\sqrt{\Delta_0}}\Theta(\Delta_0)\Theta(1-c_2^2) \quad . \tag{5.36}$$

In doing so, one easily notes that $c_1 + x\cos\delta$ vanishes only at points where $\Delta_1 \leq 0$, so that the sgn term can be treated effectively as a constant. The real part can

be computed from [14], integral No. 231.10. Bearing the definition of $x_{1/2}$ from Eq. (5.32) in mind, one obtains

$$\mathcal{P}\int_{-1}^{+1} \frac{\Theta(\Delta_1)\mathrm{sgn}(c_1 + x\cos\delta)}{\sqrt{\Delta_1}(x + c_2)} dx$$

$$= \frac{\Theta(\Delta_0)}{\sqrt{\Delta_0}} \Big[\mathrm{sgn}(c_1 + \cos\delta)\Big[F(+1) - \Theta(1 - c_1^2)F(x_2)\Big]$$

$$-\mathrm{sgn}(c_1 - \cos\delta)\Big[F(-1) - \Theta(1 - c_1^2)F(x_1)\Big]\Big]$$

$$+\frac{\Theta(-\Delta_0)}{\sqrt{-\Delta_0}} \Big[\mathrm{sgn}(c_1 + \cos\delta)\Big[G(+1) - G(x_2)\Big]$$

$$-\mathrm{sgn}(c_1 - \cos\delta)\Big[G(-1) - G(x_1)\Big]\Big] \tag{5.37}$$

where

$$G(x) = \arccos \frac{(c_2 - c_1\cos\delta)(x + c_2) - \Delta_0}{\sin\delta\sqrt{1 - c_1^2}|x + c_2|} \tag{5.38}$$

and F is either one of the functions

$$F_\pm(x) = \pm\log\left|\frac{(x + c_2)(c_1\cos\delta - c_2) + \Delta_0 \mp \sqrt{\Delta_0\Delta_1}}{x + c_2}\right| \quad . \tag{5.39}$$

Eq. (5.37) can be further simplified by

$$F_\pm(x_1) = F_\pm(x_2) = \pm\log\left(\sin\delta\sqrt{1 - c_1^2}\right) \tag{5.40a}$$

$$G(x_1) = G(x_2) = 0 \quad . \tag{5.40b}$$

Up until this point, the calculation in this section has been purely analytical. However, although F_+ and F_- are equivalent in specifying the indefinite integral pertinent to Eq. (5.36), they are not equally well behaved numerically if either $|c_1|$ or $|c_2|$ is approximately equal to one. It is therefore necessary at this point to devise a numerically stable procedure, and we give our algorithm here. To this end, it is useful to define Ξ as

$$\Xi = \mathrm{sgn}(c_1 + \cos\delta)\Big[F(+1) - \Theta(1 - c_1^2)F(x_2)\Big]$$

$$- \mathrm{sgn}(c_1 - \cos\delta)\Big[F(-1) - \Theta(1 - c_1^2)F(x_1)\Big] \quad . \tag{5.41}$$

Using the identity

$$F_+(x) - F_-(x) = \log\left(\sin^2\delta\left|1 - c_1^2\right|\right) \quad , \tag{5.42}$$

one can show that the following algorithm gives the correct result:

- If $||c_2| - 1| < ||c_1| - 1|$: Compute Ξ as

$$\Xi = \mathrm{sgn}(c_1 + \cos\delta)F_-(+1) - \mathrm{sgn}(c_1 - \cos\delta)F_-(-1)$$
$$+ \Theta(\cos^2\delta - c_1^2)\mathrm{sgn}(\cos\delta)\log\left|\sin^2\delta(1 - c_1^2)\right| \tag{5.43a}$$

- If $||c_2| - 1| > ||c_1| - 1|$:
  - If $c_2 - c_1\cos\delta > 0$: Compute Ξ as

$$\Xi = \mathrm{sgn}(c_1 + \cos\delta)F_+(+1) - \mathrm{sgn}(c_1 - \cos\delta)F_-(-1)$$
$$- \Theta(c_1^2 - \cos^2\delta)\mathrm{sgn}(c_1)\log\left|\sin^2\delta(1 - c_1^2)\right| \tag{5.43b}$$

  - If $c_2 - c_1\cos\delta < 0$: Compute Ξ as

$$\Xi = \mathrm{sgn}(c_1 + \cos\delta)F_-(+1) - \mathrm{sgn}(c_1 - \cos\delta)F_+(-1)$$
$$+ \Theta(c_1^2 - \cos^2\delta)\mathrm{sgn}(c_1)\log\left|\sin^2\delta(1 - c_1^2)\right| \tag{5.43c}$$

This is implemented in the numerical procedure.

3. Putting the Parts together

The final expression for $\tilde{C}_0^\pm$ is now given. We may write

$$\tilde{C}_0^\pm = \frac{1}{2kq}\int_m^{\Lambda_E} dE\,\frac{f(\pm E - \mu)}{p\sqrt{|\Delta_0|}}\bigg(\Theta(\Delta_0)\Xi - \Theta(-\Delta_0)\pi\mathrm{sgn}(\lambda_1\lambda_2) \tag{5.44}$$
$$+\Theta(-\Delta_0)\Big(\mathrm{sgn}(c_1 + \cos\delta)G(+1) - \mathrm{sgn}(c_1 - \cos\delta)G(-1)\Big)$$
$$+i\pi\Theta(\Delta_0)\Big(\mathrm{sgn}(\lambda_1)\Theta(1 - c_1^2)\mathrm{sgn}(c_2 - c_1\cos\delta)$$
$$+\mathrm{sgn}(\lambda_2)\Theta(1 - c_2^2)\mathrm{sgn}(c_1 - c_2\cos\delta)\Big)\bigg) \quad ,$$

where Ξ has to be taken from Eq. (5.43). In the spirit of Sec. IV D, in order to evaluate this integral, one has to identify the singular points of the integrand. It is easy to see that the following points are singularities:

- Energies, at which $|c_1| = 1$ or $|c_2| = 1$. These can be calculated from Eq. (4.20).
- Energies, at which $\Delta_0 = 0$. To compute these energies, one notes that c_1 and c_2 can be written as

$$c_1 = \frac{a_1 + b_1 E}{p} \qquad\qquad c_2 = \frac{a_2 + b_2 E}{p} \quad , \tag{5.45}$$

  where a_i and b_i can be obtained from Eq. (5.26). With these constants, the equation $\Delta_0 = 0$ can be cast into the form

$$\begin{aligned} 0 = {} & E^2(b_1^2 + b_2^2 - 2b_1 b_2 \cos\delta - \sin^2\delta) \\ & + 2E(a_1 b_1 + a_2 b_2 - (a_1 b_2 + a_2 b_1)\cos\delta) \\ & + (a_1^2 + a_2^2 - 2a_1 a_2 \cos\delta + m^2 \sin^2\delta) \end{aligned} \tag{5.46}$$

  which can be easily solved for E.
- Energies, at which $|c_1| = |\cos\delta|$. These can be computed by

$$E = \frac{1}{2(k^2\cos^2\delta - \lambda_1^2)} \left(\lambda_1 a_1 \pm k\cos\delta\sqrt{a_1^2 - 4m^2(\lambda_1^2 - k^2\cos^2\delta)}\right) \quad . \tag{5.47}$$

All of these points correspond to integrable singularities. On closer inspection, one also obtains that the energies, at which the arguments of the logarithmic function (in $F_\pm$) vanish, or the energies, at which the argument of the arccosine (in G) becomes greater than one, are included in these cases.

To illustrate C_0 for the general case, we plot $m^2 C_0$ in Fig. 7 as a function of k_0/Λ with $k = 3/4k_0$, $q = 3/4q_0$ and $\delta = \pi/6$. All other parameters are as in Fig. 5.

VI. APPLICATION: KAON MASSES IN THE NJL MODEL

In this section, we give a specific example for a calculation using our elementary integrals. It is also intended for the benefit of our reader, who wishes to compare

his/her results with our numbers. To this end, we compute the kaon masses in the framework of the $SU(3)$ Nambu–Jona–Lasinio model [3,4,9] as a function of temperature. In this model, the quark masses have to be determined from the coupled gap equations

$$m_q = m_{0q} + 4GN_c i\mathrm{tr}_\gamma S^q(x,x) + 2KN_c^2(i\mathrm{tr}_\gamma S^q(x,x))(i\mathrm{tr}_\gamma S^s(x,x)) \tag{6.1a}$$

$$m_s = m_{0s} + 4GN_c i\mathrm{tr}_\gamma S^s(x,x) + 2KN_c^2(i\mathrm{tr}_\gamma S^q(x,x))(i\mathrm{tr}_\gamma S^q(x,x)) \tag{6.1b}$$

with the current quark masses m_{0q}, m_{0s}, the number of colors N_c and the coupling constants G and K. S^f denotes the finite temperature quark propagator

$$S^f(\vec{x}-\vec{x}',\tau-\tau') = \frac{i}{\beta}\sum_n e^{-i\omega_n(\tau-\tau')}\int\frac{d^3p}{(2\pi)^3}\frac{e^{i\vec{p}(\vec{x}-\vec{x}')}}{\gamma_0(i\omega_n+\mu_f)-\vec{\gamma}\vec{p}-m_f} \tag{6.2}$$

The trace of the propagator can be expressed in terms of the function A:

$$i\mathrm{tr}_\gamma S^f(x,x) = -\frac{m_f}{4\pi^2}A(m_f,\mu_f) \tag{6.3}$$

and the gap equation takes the form

$$m_q = m_{0q} - \frac{N_c}{\pi^2}m_q A(m_q,\mu_q,\ \beta,\Lambda)\left(G - \frac{KN_c}{8\pi^2}m_s A(m_s,\mu_s,\ \beta,\Lambda)\right) \tag{6.4a}$$

$$m_s = m_{0s} - \frac{GN_c}{\pi^2}m_s A(m_s,\mu_s,\ \beta,\Lambda) + \frac{KN_c^2}{8\pi^4}\left(m_q A(m_q,\mu_q,\ \beta,\Lambda)\right)^2 \tag{6.4b}$$

Meson masses are computed from the quark-antiquark scattering matrix. The scattering matrix for light and strange quarks in the pseudoscalar channel can be written as

$$M = \frac{2K_{\mathrm{eff}}}{1-4K_{\mathrm{eff}}\Pi(k_0,k)} \quad , \tag{6.5}$$

where $(k_0,\vec{k})$ is the total four momentum of the quarks, $\Pi(k_0,k)$ the irreducible polarization and K_{eff} an effective coupling constant, which is given by

$$K_{\mathrm{eff}} = G + \frac{KN_c}{2}i\mathrm{tr}_\gamma S^q(x,x) = G - \frac{KN_c}{8\pi^2}m_q A(m_q,\mu_q,\ \beta,\Lambda) \tag{6.6}$$

The irreducible polarization $\Pi(k_0,k)$ is computed from the diagram in Fig. 8. One obtains

$$-i\Pi(i\nu_m,k) = -N_c\frac{i}{\beta}\sum_n\int\frac{d^3p}{(2\pi)^3}\mathrm{tr}_\gamma\left[iS^q(i\omega_n,\vec{p})i\gamma_5 iS^s(i\omega_n-i\nu_m,\vec{p}-\vec{k})i\gamma_5\right]$$

$$= 4iN_c\frac{1}{\beta}\sum_n\int\frac{d^3p}{(2\pi)^3}\frac{(i\omega_n+\mu_q)(i\omega_n-i\nu_m+\mu_s)-\vec{p}(\vec{p}-\vec{k})-m_q m_s}{\left[(i\omega_n+\mu_q)^2-E_q^2\right]\left[(i\omega_n-i\nu_m+\mu_s)^2-E_s^2\right]} \tag{6.7}$$

where $E_q = \sqrt{p^2+m_q^2}$ and $E_s = \sqrt{(\vec{p}-\vec{k})^2+m_s^2}$. To cast this into a form which contains the functions A and B_0, one uses the identity

$$\begin{aligned}&(i\omega_n+\mu_q)(i\omega_n-i\nu_m+\mu_s)-\vec{p}(\vec{p}-\vec{k})-m_q m_s\\ &\quad=\frac{1}{2}\Big[\Big((i\omega_n+\mu_q)^2-E_q^2\Big)+\Big((i\omega_n-i\nu_m+\mu_s)^2-E_s^2\Big)\\ &\qquad+\Big((m_q-m_s)^2-(\mu_q-\mu_s+i\nu_m)^2+k^2\Big)\Big]\end{aligned} \tag{6.8}$$

After continuing $i\nu_m$ to k_0, one immediately obtains the result

$$\begin{aligned}\Pi(k_0,k) = -\frac{N_c}{8\pi^2}\Big[&A(m_q,\mu_q,\ \beta,\Lambda)+A(m_s,\mu_s,\ \beta,\Lambda)\\ &+\Big((m_q-m_s)^2-(k_0+\mu_q-\mu_s)^2+k^2\Big)B_0(k,\ m_q,\mu_q,\ m_s,\mu_s,k_0,\ \beta,\Lambda)\Big]\quad.\end{aligned} \tag{6.9}$$

The kaon mass is now computed using the dispersion relation

$$1-4K_{\mathrm{eff}}\Pi(m_K,0)=0\quad. \tag{6.10}$$

Using the numbers $m_{0q} = 5.5$MeV, $m_{0s} = 140.7$MeV, $\mu_q = \mu_s = 0$, $\Lambda = 602.3$MeV, $G\Lambda^2 = 1.835$ and $K\Lambda^5 = 12.36$, one obtains the zero temperature results $m_q = 367.7$MeV and $m_s = 549.5$MeV from Eq. (6.4). Solving Eq. (6.10) with these numbers yields $m_K = 497.7$MeV. The temperature dependence of m_q, m_s and m_K is computed using the same formulae and the result is depicted in Fig. 9. Further examples can be found in Ref. [9].

VII. SUMMARY AND CONCLUSION

In this paper, we have presented the technical aspects required for a numerical evaluation of the one, two and three fermion line integrals at finite temperature, structuring each as a sum of terms of a specific generic integral taken at different

values of its arguments, that are often required for field theoretic calculations such as within the Nambu–Jona–Lasinio model. Both real and imaginary parts of these functions are explicitly calculated. Concomitantly, we have illustrated all functions graphically for certain parameter sets, in order to enable the user to verify his/her own calculation. A simple example, employing two of the integrals has also been shown. A computer source code is also available that routinizes these integrals. The generalization to bosonic line integrals can also be seen to follow in an analogous fashion.

ACKNOWLEDGMENTS

We would like to thank J. Hüfner for his encouragement and support of this project, and those many people who have requested that we make this work generally available. We are also indebted to the Bundesministerium für Bildung und Wissenschaft and the Deutsche Forschungsgemeinschaft, who have supported this project under contract numbers 06 HD 742 and Hu 233/4-3 respectively.

APPENDIX A: PROOF OF EQ. (5.28)

Our assertion is

$$\lim_{\epsilon\to 0}\int_0^{2\pi}\frac{d\phi}{a+b\cos\phi-i\epsilon}=\frac{2\pi}{\sqrt{|a^2-b^2|}}\Big(\Theta(a^2-b^2)\mathrm{sgn}(a)+i\Theta(b^2-a^2)\Big)\quad . \tag{A1}$$

The real part of this equation can be easily obtained from integral tables. (See e. g. [14], integral No. 331.41.) The singularity in the integral only appears for $|b|>|a|$, which gives the Θ function for the imaginary part. In this case, singularities appear at

$$\phi=\arccos(-a/b)\qquad\text{and}\qquad\phi=2\pi-\arccos(-a/b)\quad . \tag{A2}$$

The first of these singularities appears in the interval $[0,\pi]$, the second one in the interval $[\pi,2\pi]$. We split the integral into two parts

$$\begin{aligned}\lim_{\epsilon\to 0}\int_0^{2\pi}\frac{d\phi}{a+b\cos\phi-i\epsilon}&=\lim_{\epsilon\to 0}\frac{1}{b}\int_0^{\pi}\frac{d\phi}{a/b+\cos\phi-i\epsilon\mathrm{sgn}(b)}\\&\quad+\lim_{\epsilon\to 0}\frac{1}{b}\int_\pi^{2\pi}\frac{d\phi}{a/b+\cos\phi-i\epsilon\mathrm{sgn}(b)}\end{aligned} \tag{A3}$$

and focus on the first of these two parts. Let $\phi_0 = \arccos(-a/b)$ and the function $f(\phi)$ be defined by

$$f(\phi) = \begin{cases} (\cos\phi + a/b)/(\phi - \phi_0) & \text{for} \quad \phi \neq \phi_0 \\ \\ \sin\phi_0 = \sqrt{1-(a/b)^2} & \text{for} \quad \phi = \phi_0 \end{cases} \quad . \tag{A4}$$

It can be easily seen that $f(\phi)$ is continous in the interval $[0, \pi]$ and

$$\begin{aligned} \lim_{\epsilon \to 0} \frac{1}{b} \int_0^\pi \frac{d\phi}{a/b + \cos\phi - i\epsilon \mathrm{sgn}(b)} &= \lim_{\epsilon \to 0} \frac{1}{b} \int_0^\pi \frac{d\phi}{(\phi - \phi_0) f(\phi) - i\epsilon \mathrm{sgn}(b)} \\ &= \lim_{\epsilon \to 0} \frac{1}{b} \int_0^\pi \frac{d\phi}{(\phi - \phi_0 - i\epsilon \mathrm{sgn}(b)) f(\phi)} \end{aligned} \tag{A5}$$

since $f(\phi_0)$ is positive. Our standard formula Eq. (4.12) can be applied to this integral to yield

$$\Im \left(\lim_{\epsilon \to 0} \frac{1}{b} \int_0^\pi \frac{d\phi}{a/b + \cos\phi - i\epsilon \mathrm{sgn}(b)} \right) = \frac{1}{b} \pi \mathrm{sgn}(b) \frac{1}{f(\phi_0)} = \frac{\pi}{\sqrt{b^2 - a^2}} \quad . \tag{A6}$$

The second integral on the right hand side of Eq. (A3) can be treated in the same fashion and is found to give the same contribution. QED.

APPENDIX B: PROOF OF EQ. (5.33)

The assertion is

$$\lim_{\epsilon \to 0} \int_a^b \frac{dx}{\sqrt{(b-x)(x-a)}(x + c - i\epsilon)} = \frac{\pi}{\sqrt{|\Delta|}} \left(\overline{\mathrm{sgn}(x+c)} \Theta(\Delta) + i\Theta(-\Delta) \right) \tag{B1}$$

with $\Delta = (a+c)(b+c)$. To prove this, we first apply Eq. (4.12) to obtain

$$\lim_{\epsilon \to 0} \int_a^b \frac{dx}{\sqrt{(b-x)(x-a)}(x + c - i\epsilon)} = \mathcal{P}\!\!\int_a^b \frac{dx}{\sqrt{(b-x)(x-a)}(x + c)} + \frac{i\pi\Theta(-\Delta)}{\sqrt{-\Delta}} \quad . \tag{B2}$$

The Θ function on the right hand side of Eq. (B2) emerges from the fact that the singularity occurs only if

$$a \leq -c \leq b \quad \Leftrightarrow \quad \Delta \leq 0 \quad . \tag{B3}$$

The real part of Eq. (B2) can be looked up in integral tables. (See e. g. [13], integral No. 221.7.) For $\Delta > 0$, one obtains

$$\int \frac{dx}{\sqrt{(b-x)(x-a)}(x+c)} = \frac{1}{\sqrt{\Delta}} \arcsin\left(\frac{-(c+a)(b-x)+(c+b)(x-a)}{(b-a)|x+c|}\right) \quad \text{(B4)}$$

which gives

$$\int_a^b \frac{dx}{\sqrt{(b-x)(x-a)}(x+c)} = \frac{\pi\overline{\mathrm{sgn}(x+c)}}{\sqrt{\Delta}} \quad . \quad \text{(B5)}$$

For $\Delta < 0$, one obtains

$$\int \frac{dx}{\sqrt{(b-x)(x-a)}(x+c)} = \frac{1}{\sqrt{-\Delta}} \log \frac{\left(\sqrt{(c+a)(b-x)} - \sqrt{-(c+b)(x-a)}\right)^2}{|x+c|} \quad \text{(B6)}$$

which gives

$$\mathcal{P}\int_a^b \frac{dx}{\sqrt{(b-x)(x-a)}(x+c)} = 0 \quad . \quad \text{(B7)}$$

Combining Eqs. (B2), (B5) and (B7), proves the assertion.

REFERENCES

[1] T. Matsubara, Prog. Theor. Phys. **14**, 351 (1955).

[2] See, e. g., A. L. Fetter and J. D. Walecka, *Quantum Theory of Many Particle Systems* (McGraw–Hill, New York, 1971); A. A. Abrikosov, L. P. Gor'kov and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Prentice–Hall, Engelwood Cliffs, N. J., 1963)

[3] S. P. Klevansky, Rev. Mod. Phys. **64**, 649 (1992).

[4] T. Hatsuda and T. Kunihiro, Phys. Rep. **232**, 221 (1994).

[5] J. Hüfner, S. P. Klevansky, P. Zhuang and H. Voss, Ann. Phys. (NY) **234**, 225 (1995).

[6] P. Zhuang, J. Hüfner and S. P. Klevansky, Nucl. Phys. A **576**, 525 (1994).

[7] P. Zhuang, J. Hüfner, S. P. Klevansky and L. Neise, Phys. Rev. D **51**, 3728 (1995)

[8] J. Hüfner, S. P. Klevansky, E. Quack and P. Zhuang, Phys. Lett. B **337**, 30 (1994).

[9] P. Rehberg, S. P. Klevansky and J. Hüfner, *Hadronization in the $SU(3)$ Nambu–Jona–Lasinio Model*, Heidelberg Preprint HD–TVP–95–06, `hep-ph/9506436`.

[10] See, e. g., G. Passarino and M. Veltman, Nucl. Phys. **B** 160, 151 (1979); G. J. van Oldenborgh and J. A. M. Vermaseren, Z. Phys. C **46**, 425 (1990) and references cited therein.

[11] W. H. Press, B. P. Flannery, S. A. Teukolsky and W .T. Vetterling, *Numerical Recipes* (Cambridge University Press, 1992).

[12] R. Piessens, E. deDoncker–Kapenga, C. Überhuber and D. Kahaner, *Quadpack: a Subroutine Package for Automatic Integration* (Springer Series in Computational Mathematics, Vol. 1, 1983).

[13] W. Gröbner and N. Hofreiter, *Integraltafel; Erster Teil: Unbestimmte Integrale* (Springer Verlag, Wien, 1961).

[14] W. Gröbner and N. Hofreiter, *Integraltafel; Zweiter Teil: Bestimmte Integrale* (Springer Verlag, Wien, 1961).

FIGURES

FIG. 1. A/Λ^2 as a function of m/Λ at $\mu = 0$, $\beta \to \infty$ and $\Lambda = 602.3$MeV.

FIG. 2. B_0 for $k = 0$, $\mu_1 = \mu_2 = 0$, $m_1 = m_2 = 367.7$MeV, $T = 0$ and $\Lambda = 602.3$MeV. The solid line gives the real part, the dashed line the imaginary part.

FIG. 3. B_0 for $k = 100$MeV, $\mu_1 = \mu_2 = 0$, $m_1 = m_2 = 367.7$MeV, $T = 0$ and $\Lambda = 602.3$MeV. The solid line gives the real part, the dashed line the imaginary part.

FIG. 4. B_0 for $k_0 = 0$, $\mu_1 = \mu_2 = 0$, $m_1 = m_2 = 367.7$MeV, $T = 0$ and $\Lambda = 602.3$MeV. Note that B_0 is a real function in this case.

FIG. 5. $m^2 C_0$ for $k = q = 0$, $m_1 = m_2 = m_3 = 367.7$MeV, $\mu_1 = \mu_2 = \mu_3 = 0$, $q_0 = 2k_0$, $T = 0$ and $\Lambda = 602.3$MeV. The solid line gives the real part, the dashed line the imaginary part.

FIG. 6. $m^2 C_0$ for $k = 0$, $q = q_0 = 2k_0$, $m_1 = m_2 = m_3 = 367.7$MeV, $\mu_1 = \mu_2 = \mu_3 = 0$, $T = 0$ and $\Lambda = 602.3$MeV. The solid line gives the real part, the dashed line the imaginary part.

FIG. 7. $m^2 C_0$ for $k = 3/4k_0$, $q = 3/4q_0$, $\delta = \pi/6$, $m_1 = m_2 = m_3 = 367.7$MeV, $\mu_1 = \mu_2 = \mu_3 = 0$, $T = 0$ and $\Lambda = 602.3$MeV. The solid line gives the real part, the dashed line the imaginary part.

FIG. 8. Feynman diagram for the irreducible polarization function.

FIG. 9. Temperature dependence of m_q (solid line), m_s (dashed line) and m_K (dot–dashed line).

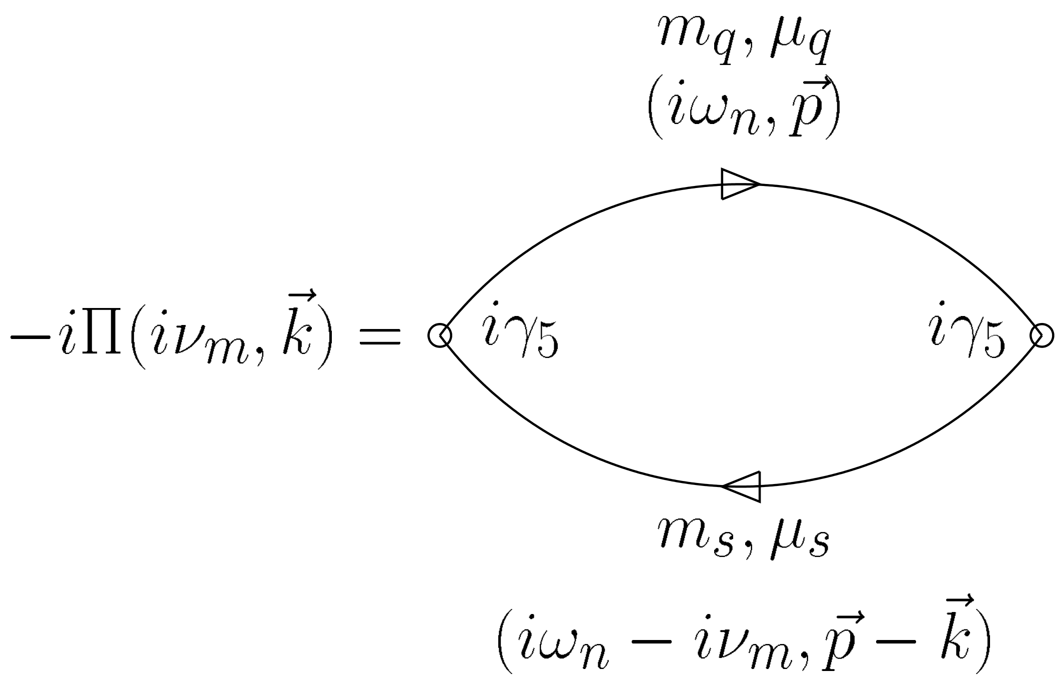

$-i\Pi(i\nu_m, \vec{k}) =$
m_q, μ_q
$(i\omega_n, \vec{p})$
$i\gamma_5$
$i\gamma_5$
m_s, μ_s
$(i\omega_n - i\nu_m, \vec{p} - \vec{k})$

Figure 8

Figure 1

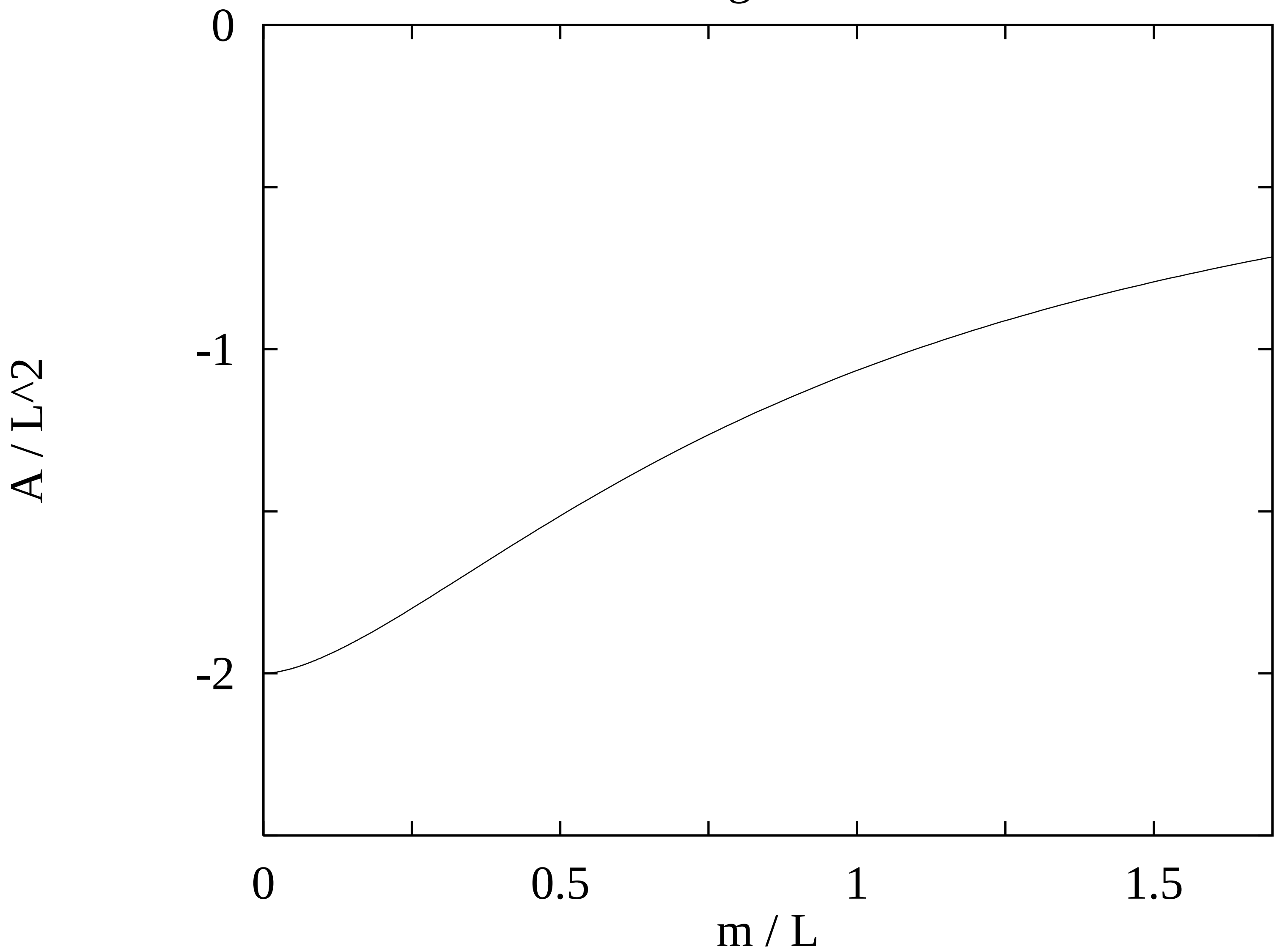


Figure 2

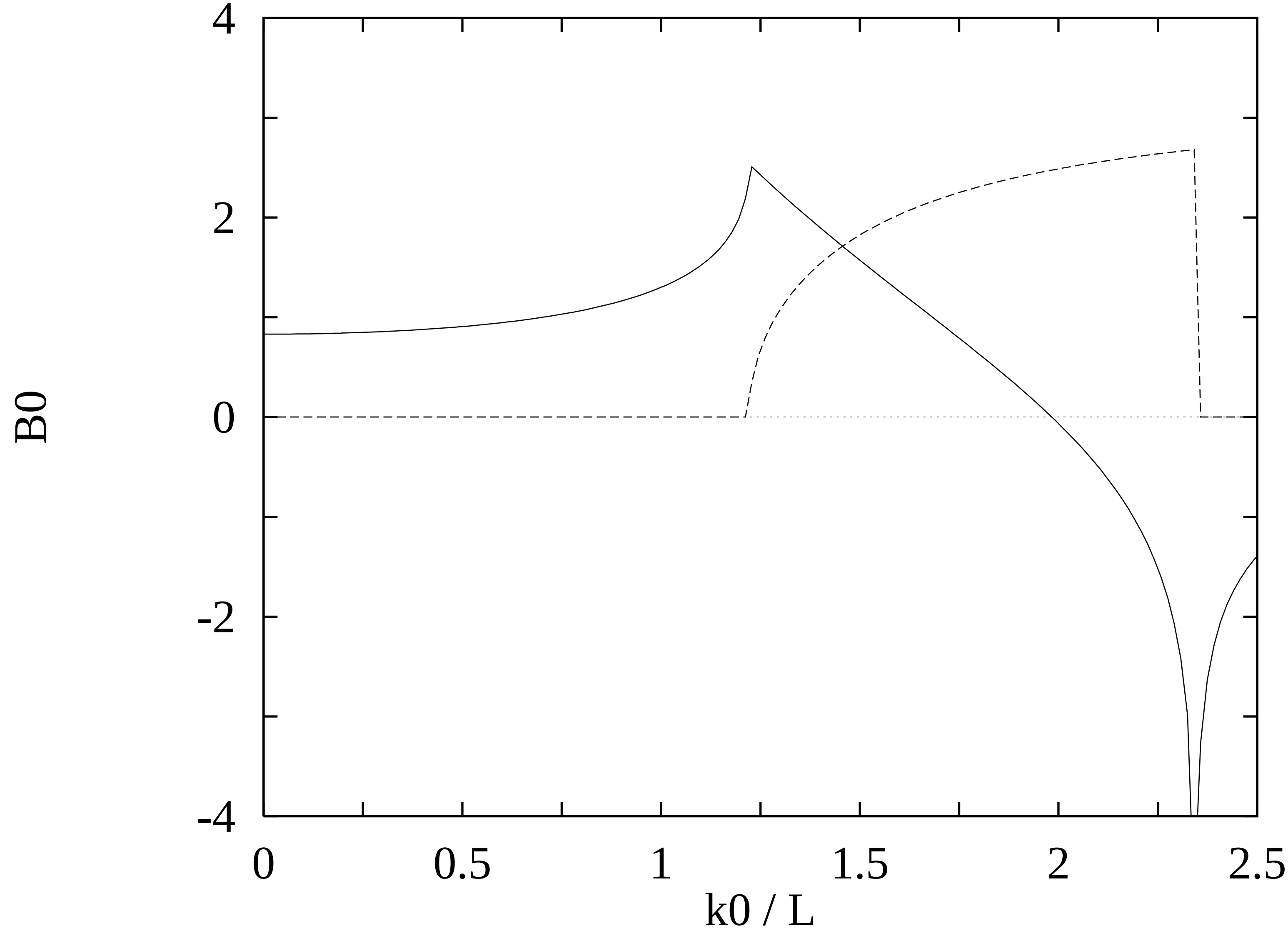


Figure 3

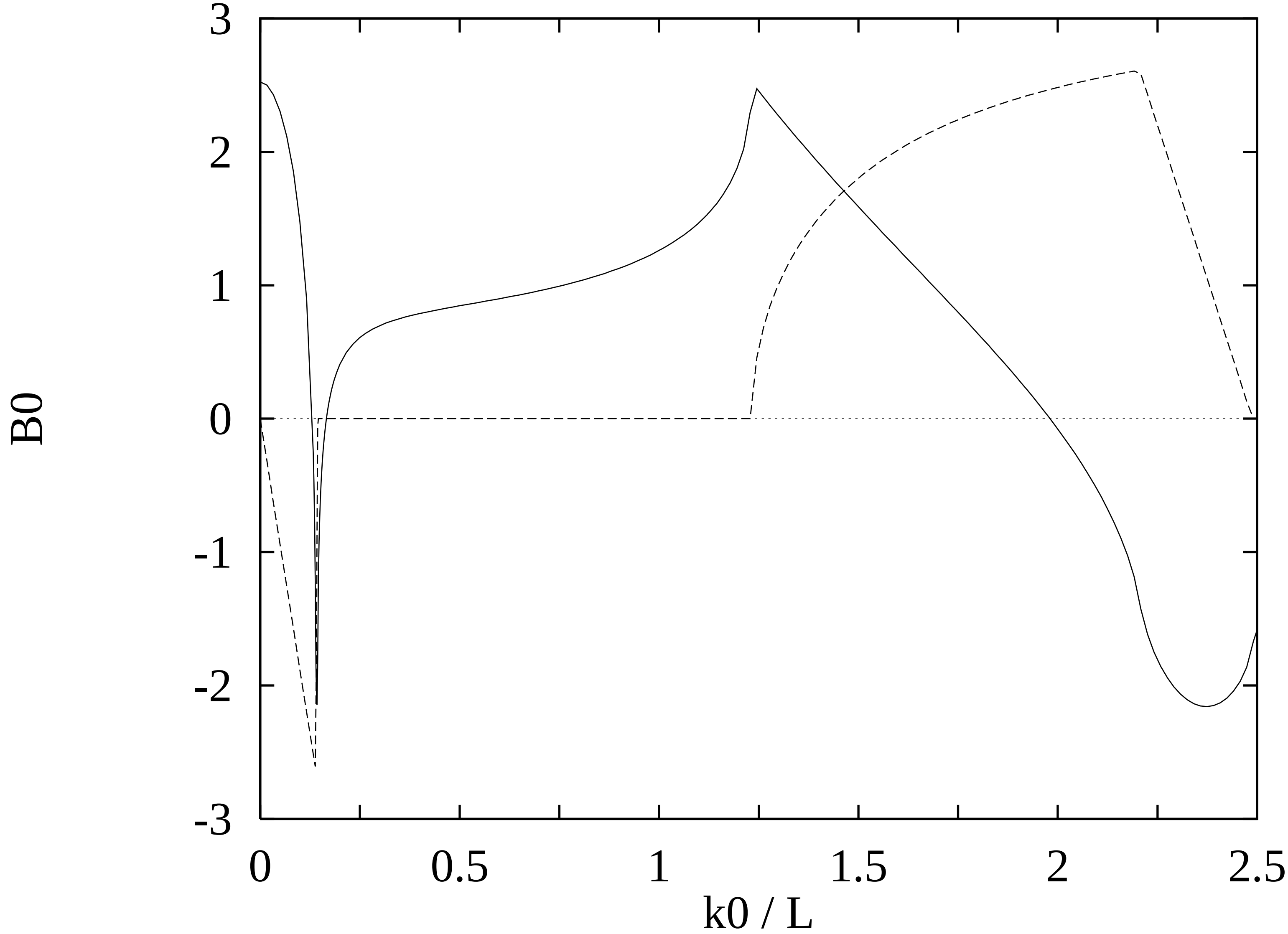


Figure 4

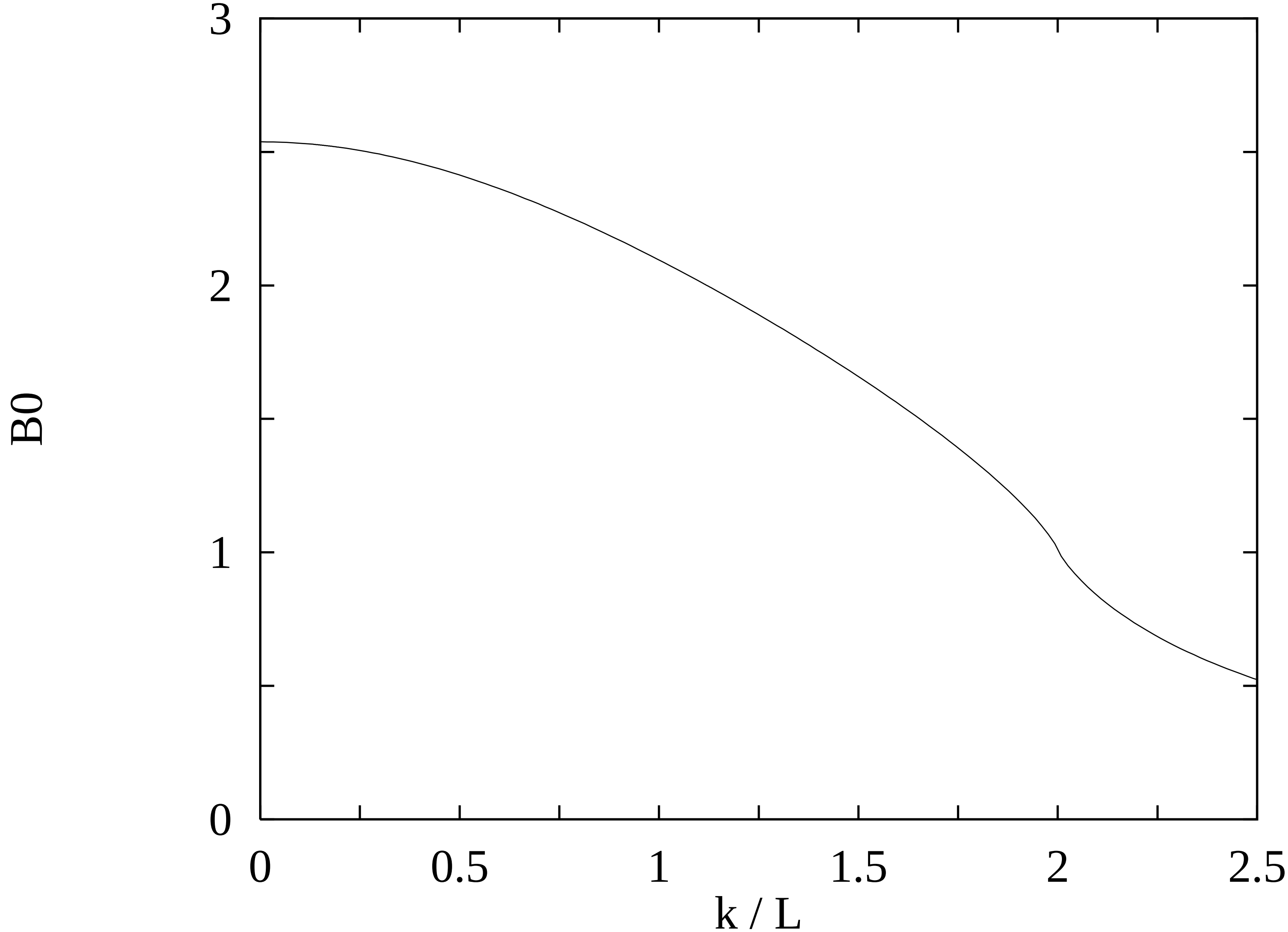
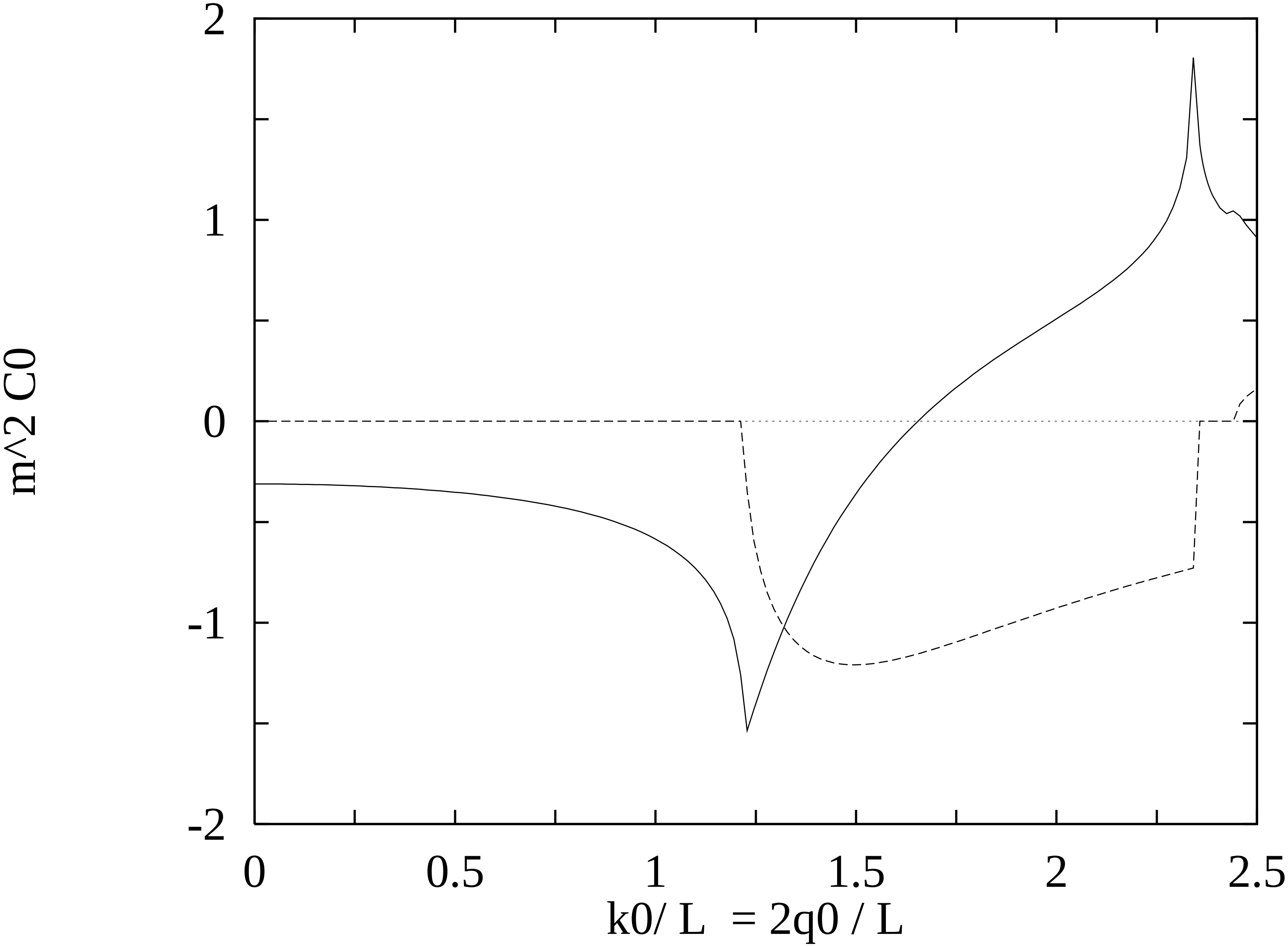


Figure 5



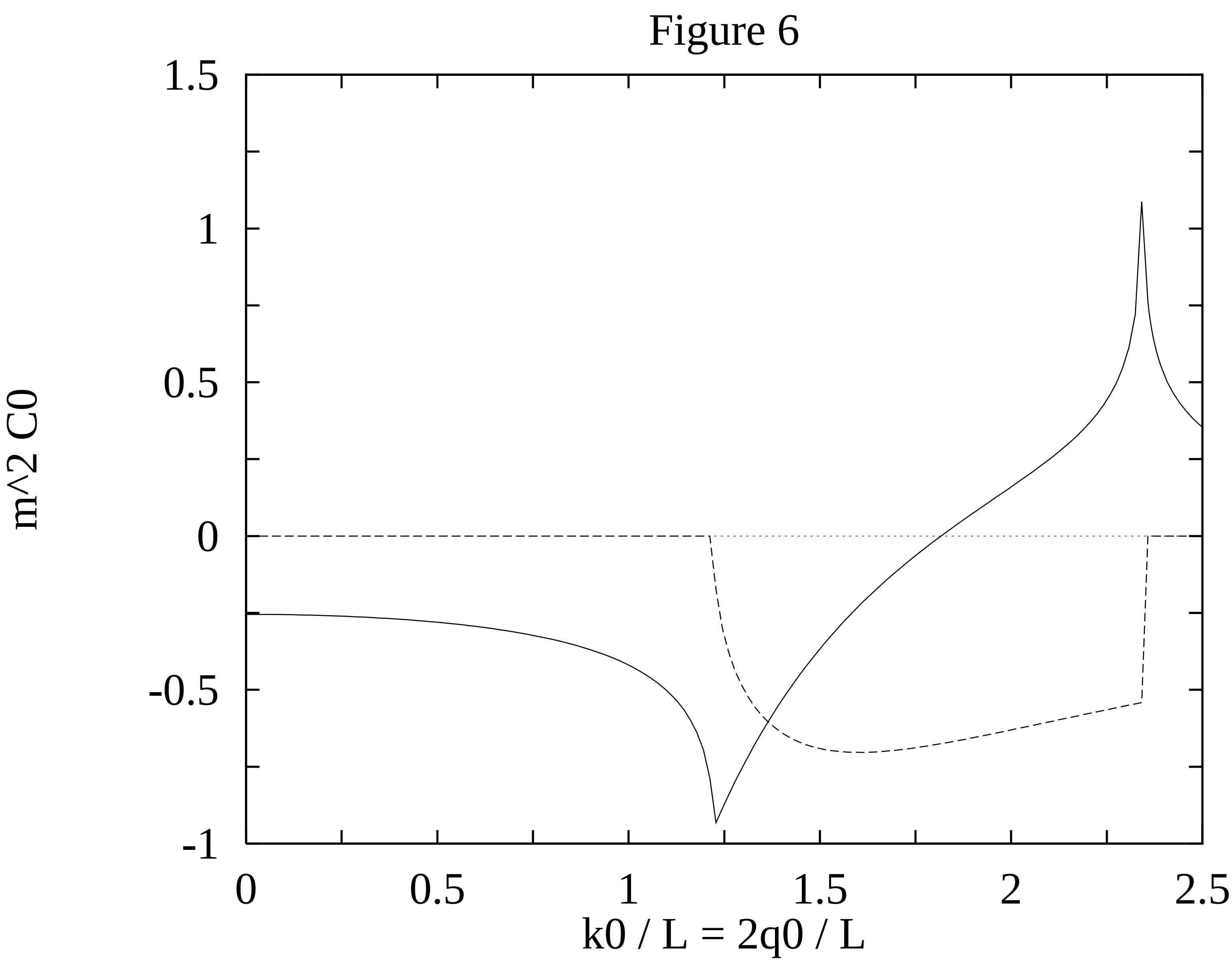


Figure 6
1.5
1
0.5
0
-0.5
-1
0
0.5
1
1.5
2
2.5
m^2 C0
k0 / L = 2q0 / L

Figure 7

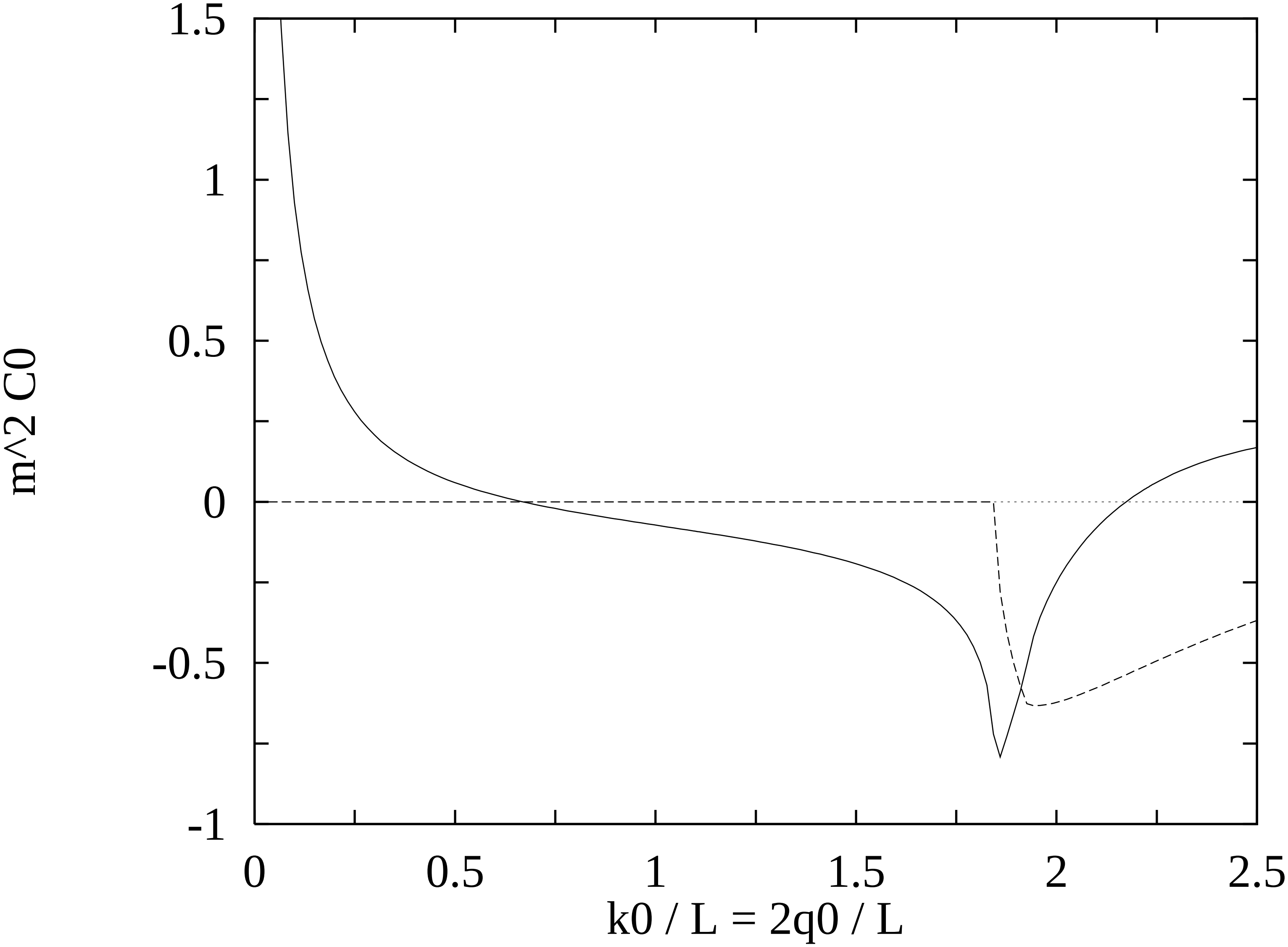


Figure 9